THE THEORY OF RELATIVITY

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THE THEORY OF RELATIVITY

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PREFACE

This introduction to the Theory of Relativity is based in part upon a course of lectures delivered in University College, London, 1912-13. The treatment, however, has been made much more systematical, and the subject matter has been extended very considerably; but, throughout, the attempt has been made to confine the reader's attention to matters of prime importance. With this aim in view, many particular problems even of great interest have not been touched upon. On the other hand, it seemed advantageous to trace the connexion of the modern theory with the theories and ideas that preceded it. And the first three chapters, therefore, are devoted to the fundamental ideas of space and time underlying classical physics, and to the electromagnetic theories of Maxwell, Hertz-Heaviside and Lorentz, from the last of which Einstein's theory of relativity was directly derived. In the exposition of the theory itself free use has been made not only of the matrix method of representation employed by Minkowski, but even more of the language of quaternions. Very little indeed of these mathematical methods is required to follow the exposition, and this little is given in Chapter V., in a form which will be at once accessible to those acquainted with the elements of the ordinary vector algebra.

It is hoped that the book will give the reader a good insight into the spirit of the theory and will enable him easily to follow the more subtle and extended developments to be found in a large number of special papers by various investigators.

I gladly take the opportunity of expressing my thanks to Mr. William Francis and Dr. T. Percy Nunn for their kindness in reading a large portion of the MS., to Prof. Alfred W. Porter, F.R.S., for reading all the proofs and for many valuable suggestions, and to the Publishers and the Printers for the care they have bestowed on my work.

L. S.

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CORRIGENDA

Page 60, line 4 from the foot, *for* on *read* ou
,, 65, line 8, *for* P *read* P'
,, 71, ,, 4, *for* $\mathfrak{b}/c^2$ *read* $\mathfrak{b}^2/c^2$
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,, 222, line 2 from the foot, *for* vectors *read* vector operators
,, 229, line 13, interchange the words 'real' and 'imaginary'
,, 235, ,, 20, *for* $\mathbf{P}'\mathbf{p}'$ *read* $(\mathbf{P}'\mathbf{p}')$
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CHAPTER I.

CLASSICAL RELATIVITY.

BEFORE entering upon the subject proper of this volume, namely, the modern doctrine of Relativity and the history of its origin and development, it seems desirable to dwell a little on the more familiar ground of what might be called the *classical* relativity, and to consider two particular points which are of fundamental importance, not only for the appreciation of the whole subject to follow, but also for an adequate understanding of almost all physico-mathematical considerations. What I am alluding to are the following questions: 1° the choice of a framework of axes or, more generally, of *a system of reference in space*, and 2° the definition of *physical time*, or the selection of a clock or time-keeper, to be employed for the quantitative determination of a succession of physical events.

Both of these questions existed and were solved, at least implicitly, a long time before the invention of the modern Principle of Relativity, in fact centuries ago, in their essence as early as Copernicus founded his system.*

The question of a space-framework is obvious enough and widely known; it will require therefore only a few simple remarks.

The most superficial observation of everyday life would suffice to show that the form and the degree of simplicity of the statement of the laws of physical phenomena, more especially of the laws of motion of what are called material bodies, depend essentially on our selection of a system of reference in space. Certain frameworks of reference are peculiarly fitted for the representation of a particular

*A clear and beautiful statement of the fundamental importance of the Copernican idea is to be found in P. Painlevé's article 'Mécanique' in the collective volume *De la Méthode dans les Sciences*, edited by Émile Borel. (Paris, F. Alcan, 1910.)

instance of motion of a particular body or also of almost any observable motion of bodies in general, leading to a high degree of completeness, exactness and simplicity, while other frameworks (moving in an arbitrary manner relatively to those) give of the same phenomena a most complicated, intricate and confused picture.*

Suppose that somebody, ignorant of the work of Copernicus, Galileo and Newton, but otherwise gifted with the highest experimental abilities and mathematical skill (a quite imaginary supposition, being hardly consistent with the first one), chooses the interior of an old-fashioned coach, driven along a fairly rough road, as his laboratory and tries to investigate the laws of motion of bodies enclosed together with him in the coach—say, of a pendulum or of a spinning top—and selects that vehicle as his system of reference. Then his tangible bodies and his conceptual 'material points,' starting from rest or any given velocity, would describe the most wonderful paths, in incessant shocks and jerky motions; the axis of *his* 'free gyroscope' would oscillate in a most complicated way,—never disclosing to him the constancy of the vector known to us as the 'angular momentum,' *i.e.* the rotatory analogue of Newton's first law of motion. Nor would the uniform translational motion have for him any peculiarly simple or generally noteworthy properties at all. His mechanical experience being, in a word, full of surprises, he would soon give up his task of stating any laws of motion whatever with reference to the coach. Getting out of it on to firm ground, he will readily find out that the earth is a much better system of reference. With this framework, smoothness and simplicity will take the place of hopeless irregularity. Undoubtedly, this property must have been remarked in a very early stage of man's history, and the above example will appear to the least trained student of mechanics of our present times trivial and simply ridiculous. 'Of course,' he would say, 'the motions of material bodies relatively to that coach are so very complicated, for that vehicle is itself moving in a highly complicated way.' He would hardly consider it worth while to add 'relatively to the earth.' The coach being such a small, insignificant thing in comparison with the terrestrial globe, it would seem extravagant to our interlocutor, if somebody insisted rather on saying that it is the earth which moves in such a complicated way relatively

* And as to 'absolute motion,' regardless of *any* system of reference, it is needless to mention that it is devoid of meaning in exactly the same way as 'absolute position.'

to the coach on its particular journey. But, as a matter of fact, both of these reference-systems move relatively *to one another*, and the comparative insignificance of one of them would, by itself, be but a very feeble argument (as we shall see presently, from another example).

At any rate the earth, the 'firm ground,' allowance being made for occasional large shocks and for very small but incessant oscillations of every part of its surface,* has proved to be an excellent system of reference for almost all motions, especially those on a small scale with regard to space and time, and practically without any reservation for all pieces of machinery and technical contrivance. In fact, the earth as a system of reference offered at once the advantage of a high degree of simplicity of description of states of equilibrium and motion, opening a wide field for the application of Newton's mechanics, at least as regards purely terrestrial observations and experiments.† The earth is then a reference-system which is constantly used also by the most advanced modern student of mechanics.

But things become altogether different when we look up to the sky and desire to bring into our mechanical scheme also the motions of those luminous points, the celestial bodies, including, of course, our satellite, the moon, and our sun. Then the earth loses its privilege as a framework of reference. If it were only for the so-called 'fixed stars,' which form the enormous majority of those luminous points (and the moon too), we could still satisfy our vanity and continue to consider our globe as an universal mechanical system of reference, *the* system of reference, as it were. On our plane drawings, or in our three-dimensional models, we could then represent the earth by a fixed disc, or sphere, respectively, with a smaller sphere moving round it in a circular orbit, to imitate our moon, the whole surrounded by a large spherical shell of glass sown with millions of tiny stars, spinning gently and uniformly round the earth's axis,—very

* Which gave so much trouble to the late Sir G. H. Darwin and his brother in their attempts to measure directly the gravitational action of the moon, as described in Sir G. H. Darwin's attractive popular book, *The Tides and Kindred Phenomena in the Solar System*, London, 1898 (German edition by A. Pockels, enlarged; Teubner, Leipzig, 1911).

† With the exception of those of the type of Foucault's pendulum experiments, performed with the special purpose 'of showing the earth's rotation.' In more recent times the pendulum could be successfully replaced by a gyroscope, as originally suggested, and tried, by Foucault himself.

much like, in fact, some primitive mental pictures of the universe.* But the case becomes entirely different when we come to consider the far less numerous class of luminous points or little discs, the planets, and the comets, moving visibly among the 'fixed' shining points in a complicated way. Then, even before touching any dynamical part of the celestial problem, we are compelled to give up our earth as a system of reference and replace it by that of the 'fixed stars,' originally so inconspicuous, or—what turns out to be equally good—by a framework of axes pointing from an initial point fixed in the sun towards any given triad of fixed stars. It is needless to tell here again the long story of that admirable and ingenious system which was founded by Ptolemy (born about 140 B.C.), which held the field during fourteen centuries, to be replaced finally and definitely by the system of Copernicus (1473-1543), which transferred to the sun the previous dignity of the earth.† The Copernican system of reference had the enormous advantage of simplicity, quite independently of any mechanical, *i.e.* (to put it more strictly) dynamical considerations. Its superiority to the geocentric system manifested itself already in the simplicity it gave to the paths of the solar family of bodies, the wonderfully simple shapes of the orbits of the planets. In the geocentric scheme we had the complicated system of 'excentrics and epicycles' of Ptolemy, whereas taking, in our drawing or model, the sun as fixed, the orbits of the planets became simple circles, which in the next step of approximation turned out to be slightly elliptic. Thus the Copernican system of reference had its enormous advantages before any properly mechanical point of the subject was entered upon. Historically, in fact, the mechanics of Galileo and Newton came a long time after Copernicus, so that the

* The earth as the centre of the universe, with the 'crystal spheres,' with the stars stuck to them, spinning round the earth, still formed part of the teachings of the Ionian school of philosophers founded by Thales (born about 640 B.C.). The first to suggest the rotation of the earth round its axis and its motion round the sun seems to have been Pythagoras, one of Thales' disciples, though it has been later unjustly attributed to Philolaus, one of Pythagoras' disciples (born about 450 B.C.).

† Although I do not claim to give here anything like a history of astronomy, it may be worth mentioning that the Pythagoreans already taught that the planets and comets were circling round the sun. But at any rate the Ptolemaean geocentric system reigned universally from the second till the fifteenth century, the only serious objection against its complexity having been raised in the thirteenth century by Alphonso X., king of Castile, the author of the astronomical 'Tables' associated with his name (published during 1248-1252).

privilege of reference-system was taken away from our earth and transferred to the sun on the ground of purely kinematical considerations of simplicity, a few centuries before Newton. But afterwards the Copernican or the 'fixed-stars' system of reference appeared to be wonderfully appropriate to Newtonian mechanics, both in its original shape and in its later (chiefly formal) development by Laplace for celestial and by Lagrange for terrestrial and general problems. It soon became the final reference-system of mechanics. It is relatively to this 'fixed-stars' system of reference that the law of inertia has proved to be valid. We will call it, therefore, following the modern habit, the *inertial system*, or sometimes, also, the *Newtonian system of reference.** It is relatively to this system that spinning bodies behave in the characteristically simple manner which has led many authors to speak of their property of 'absolute orientation.' Or, to put it in less obscure words, it is relatively to the inertial system that the vector called angular momentum is preserved, both in size and in direction,—this property being a consequence of the fundamental laws of Newton's mechanics, and, at the same time, a perfect and most instructive analogue to Newton's First Law of motion.† The most immediate and tangible manifestation of this property is that the axis of a free gyroscope (practically coinciding in direction with its angular momentum) points always towards the same fixed star; thus having the simplest relation to the inertial system, since it is invariably orientated in this system of reference. Notice that it would, therefore, be more extravagant to say that the axis of such a gyroscope moves relatively to the earth than *vice versa*,—though apparently, bodily, the gyroscope of human make is such an inconspicuous tiny thing in comparison with our planet. The conservation of the angular momentum, or moment of momentum, $\Sigma m \mathrm{V}\mathbf{rv}$,‡ of the whole solar system, which is best known in connexion with Laplace's 'invariable plane,' is but the same thing on a larger scale than that exhibited by our spinning tops. But this only by the

* We speak of it in the singular, instead of infinite plural, only for the sake of shortness. For, as is well known, if Σ, say the 'fixed' stars, be such a system, then any other system Σ' having relatively to Σ any motion of uniform (rectilinear) translation is equally good for all purposes.

† This point is expressly insisted upon and successfully applied to didactic purposes in Professor A. M. Worthington's *Dynamics of Rotation*, sixth edition, new impression 1910; Longmans, Green & Co., London.

‡ See, for example, the author's *Vectorial Mechanics*, Chap. III.; Macmillan & Co., London, 1913.

way. What mainly concerns us here is that the 'fixed-stars' system —or, more rigorously, any one out of the ∞^3 multitude of equivalent inertial systems—has gradually turned out to be peculiarly fitted as a system of reference for the representation of the motion of material bodies.

But also with this system of reference the laws of motion have their simple, Newtonian form only for a t measured in *a certain* way, *i.e.* for a certain clock or time-keeper, *e.g.* approximately the earth in its diurnal rotation, or, more exactly (in connexion with what is known as the frictional retarding effect of the tides), a time-keeper slightly different from the rotating earth. This is equivalent to defining as *equal intervals of time* those in which a body not acted on by 'external forces,' *i.e.* very distant from other bodies or otherwise suspected sources of disturbance, describes equal paths.* In maintaining the motion of such and such a body in such and such circumstances to be *uniform*, we do not make a statement, but rather are defining what we strictly mean by equal intervals of time. Selecting quite at random a different time-keeper, we could not, of course, expect the same simple laws to hold, with respect to the inertial system of reference. But with another space-framework of reference another time-keeper might do as well.

Thus we see that, to a certain extent, the choice of a system of reference in space has to be made in conjunction with the selection of a time-keeper. Our x, y, z, t, the whole tetrad, the space *and* time framework must be selected as one whole. That kind of 'union' emphasized by the late Hermann Minkowski, the joint selection of x, y, z, t, manifesting itself in the modern relativistic theory by the consideration of a four-dimensional 'world' (instead of time and space, separately), is not altogether such an entirely new and revolutionary idea as is generally believed; for to a certain extent, and in a somewhat different sense, it is as well a requirement of Newtonian mechanics, and, more generally, of the classical kind of Physics, as of modern Relativity. What difference there really is between the two we shall see in the following chapters.

* Thus it is manifest that the science of mechanics does not describe the motion of bodies in its quantitative dependence upon 'time, flowing at a constant rate' (Newton), but literally gives only sets of *simultaneous* states of motion of the various bodies, the time-keeper itself being included. What is besides contained in these sets or successions is a non-quantitative element, namely, of what is vaguely called 'before' and 'after.'

Meanwhile we have touched, in passing, the fourth variable t, and this brings us to our second point, namely, the *definition of physical time*, the selection of 'the independent variable t' of our physico-mathematical equations, but viewed more generally, and more carefully, than above, where we have touched it only incidentally.

To explain this question, of capital importance for almost every quantitative physical research, I must ask you to direct your attention to the following considerations.

Suppose we do not limit ourselves to the investigation of motion only, but are concerned with every possible kind of physical phenomena, such as conduction of heat or electricity, diffusion of liquids or gases, melting of ice, evaporation of a liquid, etc., etc., and that we propose to describe the progress of these phenomena in time, to trace their history, past and future. How are we, then, to select our time-quantity t?

First of all, we cannot, of course, take it to be Newton's 'absolute time,' which is defined, according to a quotation from Maxwell,* as follows:

'Absolute, true, and mathematical Time is conceived by Newton as flowing at a constant rate, unaffected by the speed or slowness of the motions of material things. It is also called Duration.'

For, supposing there *is* such a thing,† we do not know how to find or to construct a clock which measures this 'absolute time,' even approximately; that is to say, we have no criterion to distinguish such a clock from a 'wrong' one. And thus, certainly, we cannot use this kind of definition for physical purposes. How are we then to measure our t? Granting that the selection of a chronometer indicating our t is (at least within certain wide limits) arbitrary or free, what is the requirement on which we have to base our choice?

Now, it seems to me that the first and most general requirement, which may also be seen to be tacitly assumed in all the investigations of both the more recent and classical natural philosophers, especially physicists and astronomers, is

> that our differential equations, representing the laws of physical (and other) phenomena, *should not contain the time*, the variable t, *explicitly*,

* *Matter and Motion*, page 19.

† But, as a matter of fact, the phrase 'flowing at a constant rate' is simply meaningless.

i.e. that for any sufficiently comprehensive physical system, of which the instantaneous state is defined, say, by $p_1, p_2, \cdots p_n$, the differential equations should be of the form

$$\left.\begin{aligned} &\frac{dp_i}{dt} = f_i(p_1, p_2, \cdots p_n), \\ &i = 1, 2, \ldots n. \end{aligned}\right\} \qquad \text{(A)}$$

This requirement is also intimately connected with a certain form of what Maxwell * calls 'the General Maxim of Physical Science' and what is commonly called the Principle of Causality.

To make my above statement more intelligible to a wider circle of (non-mathematical) readers, let us consider some very simple examples which will enable us also to see the exact meaning of instantaneous '*state*' of a system and to learn to distinguish between two very important and large classes of systems: 1) *complete* or '*undisturbed*,' and 2) *incomplete* or '*disturbed*' systems.

Suppose we have a small metallic sphere,† suspended somewhere in a large dark cellar kept at constant temperature α, receiving no heat, radiant or other, from without. Suppose we heated the sphere to 100° C., which is to be $>\alpha$ (say, $\alpha = 0°$ C.), and from that instant left it to its own fate. We return to it after an hour, as measured, say, on one of our common clocks (*i.e.* rotating earth as time-keeper), and we find it has cooled down, say, to 90°. Thus:

$$\begin{array}{ll} t & \theta \\ t_0 & 100^\circ, \\ t_0 + 1\text{ h.} & 90^\circ. \end{array}\Bigg\} \quad \begin{array}{l} \therefore\ \Delta\theta = -10^\circ, \\ \text{for } \Delta t = 1 \text{ h.} \end{array}$$

Now, if we repeated the whole experiment to-morrow or next week, we should find that during one hour the fall of temperature of our suspended sphere would again be from 100° to 90°, *i.e.* $\Delta\theta = -10°$ for $\Delta t = 1$ h. We could make similar observations for any other stage of the cooling process of our little sphere (say down from 50° instead of 100°) and for other time-intervals (say ½ h. instead of 1 h.), arbitrarily small,‡ and, repeating our observations, we should find again and again the same permanency of results,—only with different values of $\Delta\theta$ for different intervals Δt and for different starting temperatures.

* *Matter and Motion*, p. 20, first paragraph of Art. xix.; see also p. 21, lines 7-11.

† '*Small*' only so as not to be obliged to consider the different temperatures of its various parts.

‡ Or practically so, at least.

Thus, *the temperature* θ *of our sphere*, placed in the specified conditions of its environment, *varies with time* (ordinary clock-time) *in a certain determinate way*, namely, so that starting from a given temperature θ, its change during a given time-interval $\Delta t = t_2 - t_1$, *is always one and the same*, that is to say, no matter *when* this happens, independently of t_1, t_2, but depending only on

$$t_2 - t_1 = \Delta t.$$

Now, such a system, *i.e.* the sphere in its above environment, I propose to call an *undisturbed* or, what for the beginning is more cautious, **complete** system. And, in this case θ being the only quantity on whose instantaneous value the whole (thermal) future history of our sphere depends, we shall say, in accordance with general use, that the instantaneous value of the temperature θ defines the instantaneous *state* of our system (α being supposed given once and for ever). In the case before us we have a one-dimensional system, which may be called also a system of one degree of freedom.*

Take the limit of the mean rate of change $\Delta\theta/\Delta t$ for $\Delta t \to 0$; then the differential equation of our simple system will be of the form

$$\frac{d\theta}{dt} = f(\theta), \qquad (1)$$

which may be read: the instantaneous time-rate of change of the temperature is a function of its instantaneous value only.† We know in this case that $f(\theta) = -h(\theta - \alpha)$ approximately, when $\theta - \alpha$ is small, where h is a positive constant; but the particular form of the function f is for our present purposes a matter of indifference.

Let us, on the other hand, consider a similar sphere suspended, say, in a window, exposed south, in a land in which the sun is wont to shine often. Then, for the same starting value θ and same Δt, the change $\Delta\theta$ will be *different* at different times of the day, *e.g.* larger from 7 till 8 a.m. than from 2 till 3 p.m., larger in winter than in summer, and so on. Now, a system such as this sphere we will call a *disturbed* system or a system 'exposed to external agents,' or better an **incomplete** system, for this concept does not presuppose the knowledge of what is meant by 'action' of one system upon another.

* Observe that n mechanical 'degrees of freedom' amount to $2n$ degrees of freedom in the sense here adopted.

† See **Note 1** at the end of the chapter.

In the present case the differential equation of our system will be of the form

$$\frac{d\theta}{dt}=g(\theta, t), \tag{2}$$

t being again measured with the ordinary (earth-)clock, and g being some function involving t in a very complicated manner.

Now, according to the above general requirement, our t-clock would be the right one, the peculiarly fitted one, for our first physical system, (1), but not for the second, (2).

By selecting a different time-keeper we might possibly convert *some* (not all) 'disturbed' into 'undisturbed' or complete systems; but then we should spoil the completeness of (1). Let us see, first of all, what other clocks we can take instead of our original one without spoiling the simple property of (1). Instead of t, take

$$T=\phi(t);$$

then (1) will be transformed into

$$\frac{d\theta}{dT}=f(\theta)/\dot{\phi}(t), \text{ say } =\psi(T)\,.\,f(\theta).$$

Thus, if the property of completeness is to be preserved, $\dot{\phi}(t)$ must be a constant, and consequently T a linear function of t, say

$$T=t_0+\alpha t,$$

amounting only to a different initial point of time-reckoning and to the choice of a different time unit.

Now (2), the equation of our second sphere, is not of the form $d\theta/dt=\psi(t)\,.\,f(\theta)$, but rather of the form

$$\frac{d\theta}{dt}=f[\theta-\alpha(t)]+G(t);$$

consequently, if we wished even (also) to sacrifice the completeness of (1), we certainly cannot transform (2) into an undisturbed or complete system, by *any* $T=\phi(t)$. Hence the moral: certain incomplete systems *cannot* be made complete by merely selecting a new clock instead of the old one, and such systems I propose to call **essentially incomplete** systems.

But suppose we had a system obeying a law of the form

$$\frac{d\theta}{dt}=-h(t)\,.\,(\theta-\alpha), \tag{3}$$

i.e. a sphere as in (1), but having a coefficient h (coefficient of what Fourier called external conduction, divided by specific thermal capacity), which due to some visible changes of the sphere's surface, such as oxidation, is *variable*, instead of being constant. Then we could represent it as a complete system by taking instead of the t-clock another clock indicating the time

$$T=\int_0^t h(t)\,dt, \text{ say } =F(t);$$

but, $F(t)$ not being a linear function of the old time, this innovation would at once spoil the completeness of (1).

At this stage we would find ourselves in face of an alternative: which of the two systems, (1) or (3), is to be saved, which is to be sacrificed? And, correspondingly: which of the two clocks, the t-clock or the T-clock is to be selected as time-keeper? If we could detect no differences between the spheres (1), (3)—besides that of their respective thermal histories—the choice would be difficult, or rather arbitrary, quite a matter of taste or caprice. But, say, the latter sphere, (3), gets oxidized, shrinks or expands, and what not, and the former, (1), remains sensibly unaffected by the process of repeated cooling and heating. Therefore, following the maxim or principle of causality, we would conserve our t-clock, best fitted for (1), and would try to convert (3) into a complete system in a different way, namely, by taking account explicitly of the oxidation of the sphere's surface, of its dilatation, and so on, *i.e.* by introducting besides θ other quantities, say, the amount m of free oxygen present in the enclosure and the radius r of the sphere, and by defining the state of the system by the instantaneous values of θ, m r.

In this way, retaining our old clock, we should have converted the originally disturbed system of one degree of freedom into a complete system of three or more degrees of freedom. As a rule, we do not reject our traditional time-keeper at once. Encountering an incomplete or disturbed system, every physicist will, first of all, try to throw the 'disturbances' on some 'external agent' rather than on his clock. He will look round him for external agents, almost instinctively following the voice of the maxim of causality, whispering to him, as Maxwell puts it (*Matter and Motion*, p. 21): 'The difference between one event and another does not depend on the mere difference of the times.' And finding nothing particularly suspect in the nearest

neighbourhood, he will look farther round, or deeper into, the system in question.

Similarly, if we amplified the system of our second example (the sphere cooling before an open window), taking in the sun varying in position, the atmosphere, and possibly a host of other things, we would obtain a larger, more comprehensive system which, though more complicated than the original one, would satisfy us as being *undisturbed*, with our old t-clock.

So it is in many other cases. Thus, we can say:

Adding to a given fragment of nature (system), which in terms of a certain t-clock behaves like a disturbed or incomplete system $(p_1, p_2, \dots p_n)$, *i.e.* obeys the equations

$$\frac{dp_i}{dt} = f\,(p_1, p_2, \cdots p_n, t), \qquad (4)$$
$$i = 1, 2, \dots n,$$

fresh fragments of nature (with the corresponding parameters $p_{n+1}, \cdots p_{n+m}$), we often obtain a new, larger,* system which, still with the same t, is undisturbed or complete:

$$\frac{dp_i}{dt} = F_i(p_1, p_2, \cdots p_n, p_{n+1}, \cdots p_{n+m}), \qquad (5)$$
$$i = 1, 2, \dots n+m.$$

In short, we *complete* the system S_n to S_{n+m}. The t, implied here, is practically the time indicated by *that* clock which proved peculiarly fitted for the description of our previous stock of experience. Thus, for example, Fourier's theory of conduction of heat was preceded by the triumphs of classical mechanics; and if asked what the t in his fundamental equation

$$\frac{\partial \theta}{\partial t} = a^2 \nabla^2 \theta$$

meant, he would, doubtless, answer that it is to be measured by the rotating earth as time-keeper, though he hardly ever stopped in his researches to consider this matter explicitly.

Thus, generally, we do not reform our traditional clocks, but we make our systems complete as in (5), by amplifying them. But

* Not necessarily larger in volume; for often we introduce new parameters by going *deeper into* the original system itself, sometimes as deep as the molecular, atomic or even sub-atomic structure, say, of a piece of matter; or being originally concerned with the thermic history only, we supplement the temperature by the pressure, volume, electric potential, and so on.

sometimes, when we think that we have made our system S_{n+m} sufficiently comprehensive, that we have exhausted all reasonably suspected material as possible 'external agents,' and when S_{n+m} nevertheless continues to behave as an incomplete system, *i.e.* when still

$$\frac{dp_i}{dt}=F_i(p_1, \dots p_{n+m}, t), \tag{6}$$

then, to make it finally complete, we decide ourselves to change our t, our traditional clock,—especially if the change required is a slight one. This procedure, of course, is possible only when the F_i's in (6) are all of the form

$$F_i=\phi(t) \,.\, H_i(p_1, \dots p_{n+m}). \tag{7}$$

Otherwise, we feel obliged to help the matter by introducing yet fresh parameters p_{n+m+1}, p_{n+m+2}, etc., and not finding real (perceivable) supplementary material round us, we introduce *fictitious* supplements, which sometimes turn out to be real afterwards, thus leading to new discoveries.

From this it is also manifest that the Principle of Causality has the true character of a maxim; though of inestimable value both in science and in everyday life, it is not a law of nature, but rather a maxim of the naturalist.

We have classical examples of both the procedures sketched above, viz. of reforming our clocks and of supplementing or amplifying a system with the view of securing its completeness. In the first place, to get rid of one of the inequalities in the motion of the moon round the earth, astronomers have had recourse to the supposition that there is a gradual slackening in the speed of the earth's rotation. Of course, they did it in connexion with the tides and with immediate regard to the fundamental principles of mechanics, implying also the law of gravitation. But at any rate, in doing so, and in declaring that the earth as a clock is losing at the rate of 8·3, or (according to another estimate) of 22 seconds per century, they gave up the earth as their time-keeper and substituted for the sidereal time t a certain function $T=\phi(t)$, slightly differing from t, as their new '*kinetic time*,' as Prof. Love calls it.* Secondly, as is widely known, the perturbations of the planet Uranus have led Adams and Le Verrier

* A. E. H. Love, *Theoretical Mechanics*, second edition, Cambridge, 1906, page 358. In connexion with our subject, the whole of Chapter XI. of Prof. Love's book may be warmly recommended to the reader.

(working independently) to complete the system by a celestial body, at first fictitious, but then, thanks to admirable calculations based on the $\frac{1}{r^2}$-law, actually discovered and called Neptune. Notice that both kinds of procedure have essentially the character of successive approximations.

Any future researches of mechanical, thermal, electromagnetic and other phenomena, either new or old ones but treated with increasing accuracy, if leading to 'disturbed' systems, obstinately withstanding the supplementing procedure (*i.e.* that consisting in the introduction of fresh parameters p_{n+1}, etc.), may oblige us to reform also the newer, slightly corrected earth-clock, to give up the 'kinetic time' of modern astronomy for a better one, more exactly fitted for the representation of a larger field of phenomena, and so on by successive approximation. It may well happen that we shall have to give up the kinetic time for the sake of the 'electromagnetic time,'—if I may so call the variable t entering in Maxwell's differential equations of the electromagnetic field.* For suppose for a moment that some future experimental investigations of high precision were to prove that the variable t in

$$\frac{\partial \mathbf{E}}{\partial t} = c \,.\, \text{curl}\, \mathbf{M}, \quad \frac{\partial \mathbf{M}}{\partial t} = -c \,.\, \text{curl}\, \mathbf{E}$$

is not proportional to the kinetic time; then the electricians would hardly give up these admirably simple and comprehensive equations; they would rather sacrifice the kinetic time. Thus, in the struggle for *completeness* of our physical universe, we shall have always to balance the mathematical theory of one of its fragments, or sides, against that of another. A great help in this struggle is to us the circumstance that, though, rigorously, all parts of what is called the universe interact with one another, yet we are not obliged to treat at once the whole universe, but can isolate from it relatively simple

* Thus we read in Painlevé's article (*loc. cit.* page 91): 'La durée d'une ondulation lumineuse correspondant à une radiation déterminée (ou quelque durée déduite d'un phénomène electrique *constant*) sera vraisemblablement la prochaine unité de temps.' This idea seems to be suggested first by Maxwell; the corresponding wave-length would at the same time be the standard of length, when the platinum '*mètre étalon*' will be given up. Thus it may happen that the 'kinetic length' (*i.e.* that based on our notion of a 'rigid' body) will be sacrificed for the benefit of an optical or 'electromagnetic length' in the same way as the 'kinetic time' may be replaced by an 'electromagnetic time.'

parts or fragments, which behave sensibly as complete systems, or are easily converted into such.

Herewith I hope to have explained to you, at least in its fundamental points, the question of selection of a time-keeper.

Thus, we know, essentially, how to measure our t, at least in or round a *given place* (taken relatively to a certain space-framework). We do not yet know what is the precise meaning of *simultaneous* events occurring in places distant from one another. But the notion of simultaneity, especially for systems moving relatively to one another, belongs to the modern Theory of Relativity, and is, in fact, a characteristic point in Einstein's reasoning. Therefore it will best be postponed until we come to treat of the principal subject of this volume.

We could now pass immediately to the history of the electromagnetic origin of the modern principle of relativity, extending from Maxwell to Lorentz. But since we already have come to touch, more than once, Newtonian or classical mechanics, let us dwell here another moment upon this subject.

Let us call Σ one of the 'inertial' systems of reference, say the system of 'fixed' stars, and let x_i, y_i, z_i be the rectangular coordinates of the i-th particle* of a material system, relatively to Σ, at the instant t. Then the Newtonian equations of motion are

$$m_i \frac{d^2x_i}{dt^2} = X_i, \text{ etc.,} \tag{8}$$

or

$$\frac{dx_i}{dt} = u_i, \quad \frac{dy_i}{dt} = v_i, \quad \frac{dz_i}{dt} = w_i,$$

$$m_i \frac{du}{dt} = X_i, \quad m_i \frac{dv_i}{dt} = Y_i, \quad m_i \frac{dw_i}{dt} = Z_i,$$

where m_i, the masses, are constant scalars belonging to the individual particles, t is the 'kinetic time' and X_i, etc., are functions of the instantaneous state of the material system, *i.e.* of the instantaneous configuration and (in the most general case) of the instantaneous velocities of the particles relatively to one another, which for certain systems may, but for a sufficiently comprehensive system do not, contain explicitly the time t. If the material system is subject to constraints, say

$$\phi = 0, \quad \psi = 0, \quad \text{etc.,}$$

* The material 'particle' may also play the part of a planet or of the sun, as in celestial mechanics.

then X_i, etc., contain, besides the components of what are called the impressed forces, also terms like

$$\lambda \frac{\partial \phi}{\partial x_i} + \mu \frac{\partial \psi}{\partial x_i} + \dots,$$

which depend only upon the *relative* positions and relative velocities of the parts of the system (*i.e.* of the mass-particles) to one another or to the surfaces or lines on which they are constrained to remain, or to the points of support or suspension entering in such constraints. Thus the bob of a pendulum is constrained to remain at a constant distance relatively to the point of suspension, the friction of a body moving on a rough surface depends on its velocity relative to that surface, and so on. Consequently, if instead of Σ any other system of reference $\Sigma'(x', y', z')$ is taken, having relatively to Σ a *purely translational, uniform, rectilinear motion*, X_i, Y_i, Z_i are not changed. And the same thing is true of the left-hand sides of the equations of motion. For, if x_i', etc., be the coordinates of the i-th particle relatively to Σ' at the instant t, and if we take, for simplicity, the axes of x', y', z' parallel to and concurrent with those of x, y, z respectively, then

$$\left.\begin{array}{l} x_i' = x_i - ut, \quad y_i' = y_i - vt, \quad z_i' = z_i - wt, \\ t' = t, \end{array}\right\} \tag{9}$$

where (u, v, w) is the constant velocity of Σ' relatively to Σ, and where the fourth equation is added to emphasize that the old time t is retained in the transformation. Consequently,

$$u_i' = \frac{dx_i'}{dt'} = \frac{dx_i}{dt} - u = u_i - u, \quad \text{etc.}$$

(and for any pair of particles $u_i' - u_j' = u_i - u_j$, etc.), and

$$\frac{du_i'}{dt'} = \frac{du_i}{dt}, \quad \frac{dv_i'}{dt'} = \frac{dv_i}{dt}, \quad \frac{dw_i'}{dt'} = \frac{dw_i}{dt},$$

which proves the statement.

Thus, the equations of motion (8), or, in vector form,

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i, \tag{8a}$$

remain unchanged by the transformation (9), or, written vectorially, by the transformation

$$\left.\begin{array}{l} \mathbf{r}_i' = \mathbf{r}_i - \mathbf{v}t, \\ t' = t, \end{array}\right\} \tag{9a}$$

where $\mathbf{v}$, the resultant of the above u, v, w, is the vector-velocity of Σ' relatively to Σ. As regards the time, we could write also $t' = at + b$ (a, b being constants), but this would amount only to a change of units and shifting of the beginning of time-reckoning.

In view of the above property, the linear transformation (9) or (9a), $\mathbf{v}$ being any *constant* vector, is called the Newtonian (and by some authors the Galileian) transformation. Thus we can say, shortly:

The equations of classical mechanics are invariant with respect to the **Newtonian transformation.** If we assume m = a const

Notice that $\mathbf{v}$ being quite arbitrary, both as regards its size (or tensor) and direction, we have in (9a) a manifold of ∞^3 transformations, and all of these form a *group* of transformations. For, if

$$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{v}_1 t; \quad t' = t,$$

and

$$\mathbf{r}_i'' = \mathbf{r}_i' - \mathbf{v}_2 t'; \quad t'' = t',$$

then

$$\mathbf{r}_i'' = \mathbf{r}_i - \mathbf{v} t; \quad t'' = t',$$

where

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2. \tag{10}$$

We shall refer sometimes to (9) or (9a) as the **Newtonian group.**

Notice the simple additive property (10), to be compared later on with a less simple property of the corresponding group in modern Relativity.

Thus, there is no unique frame of reference for classical mechanics; if the Newtonian equations of motion are strictly valid relatively to the framework Σ of the 'fixed' stars, they are equally valid relatively to any other out of the ∞^3 frameworks Σ', connected with Σ by (9), say relatively to the solar-system frame, which has relatively to Σ a uniform velocity of something like 25 kilometres per second, towards the constellation of Hercules.* Therefore, by purely internal mechanical experiment and observation, *i.e.* not looking outside to external systems, we could never distinguish the solar frame Σ' from Σ, that is to say, Σ', like Σ, does not show any anisotropy with regard to mechanical phenomena. The same remark applies, with sufficient approximation, to the earth's annual motion: it is not ascertainable by purely terrestrial *mechanical* experiments.

Physicists hoped to detect this motion which they called also 'the motion relative to the aether,' by the means of purely terrestrial

* Quoted after Painlevé, *loc. cit.* page 117.

optical or *electromagnetic* experiments,—we shall see later how unsuccessfully.

In other words, seeing that there is no unique 'kinetic' space-framework, they tried to find a unique 'optical' or 'electromagnetic' reference-system, the 'aether,' or rather to show that this wonderful medium, already invented for other purposes, was such a unique frame of reference. But the results of all experiments of this kind have been obstinately negative.

It is chiefly this which has led to the construction of the new theory of relativity.

NOTES TO CHAPTER I.

Note 1 (to page 9). To show, generally, the connexion between the integral form of the properties of a complete system, as stated in the above illustrations, and its differential form, of which eq. (1) is an example, let us consider such a system of n degrees of freedom. Let its state at any instant t be determined by

$$p_1(t),\ p_2(t), \dots p_n(t).$$

Then, $t_0=0$ being any other, say, past instant,

$$p_i(t)=P_i[p_1(0), \dots p_n(0)\,;\ t], \quad i=1,\ 2, \dots n,$$

where P_i is a symbol of an operation or a function, implying besides the 'initial' state $p(0)$ the time-*interval* $t=t-t_0$ elapsed, but independent of the choice of the initial instant. This is the finite or integral way of expressing that the system is complete. Now let $t=a$ be any particular instant and $t=c$ another instant of time, such that

$$c=a+b.$$

Then

$$p_i(c)=P_i[p_1(a), \dots p_n(a)\,;\ b]=P_i[p_1(0), \dots p_n(0)\,;\ c],$$

so that the transformations corresponding to the passage of the system from any of its states to its successive states form a *group of transformations*, t being the (only) 'parameter' of the group. Thus we can imitate Lie's general proof of his Theorem 3 (Sophus Lie, *Theorie der Transformationsgruppen*, Leipzig, 1888; Vol. I.) for this simplest case of one

parameter. Considering $p_1(0), \dots p_n(0)$, a, c as independent variables, differentiate $p_i(c)$ with respect to a; then

$$\frac{\partial p_i(c)}{\partial p_1(a)}\frac{dp_1(a)}{da}+\dots+\frac{\partial p_i(c)}{\partial p_n(a)}\frac{dp_n(a)}{da}+\frac{\partial p_i(c)}{\partial b}\frac{\partial b}{\partial a}$$
$$=\frac{\partial p_i(c)}{\partial a}=0;$$

but $\partial b/\partial a=-1$; therefore

$$\frac{\partial p_i(c)}{\partial p_1(a)}\frac{dp_1(a)}{da}+\dots+\frac{\partial p_i(c)}{\partial p_n(a)}\frac{dp_n(a)}{da}=\frac{\partial p_i(c)}{\partial b},$$
$$i=1, 2, \dots n.$$

Now $p_1(c), \dots p_n(c)$ are mutually independent; otherwise less than n quantities p would suffice for the determination of the state of the system, contrary to the supposition. Therefore the functional determinant

$$\left|\frac{\partial p_1(c)}{\partial p_1(a)}, \cdots \frac{\partial p_n(c)}{\partial p_n(a)}\right|$$

does not vanish identically, and the above system of n equations can be solved with respect to $dp_1(a)/da$, etc., leading to

$$\frac{dp_i(a)}{da}=F_i[p_1(a), \dots p_n(a); b], \quad i=1, 2, \dots n.$$

But these equations must be valid for all values of the mutually independent magnitudes b and a. Giving therefore to b any constant value, and writing t instead of a, we obtain for any t,

$$\frac{dp_i(t)}{dt}=f_i[p\ (t), \dots p_n(t)], \quad i=1, 2, \dots n,$$

and this is the *differential* form alluded to, $f_1, f_2, \dots f_n$ being functions of the instantaneous state only.

It is instructive to consider the instantaneous state of a system as a point in the n-dimensional *space*, or *domain of states* S_n, $(p_1, p_2 \dots p_n)$, and to trace in this 'space' the **lines of states**, *i.e.* the linear continua of states assumed successively by different copies (exemplars) of the system, starting from given initial states. The differential equations of these lines of states, or, as Lie calls them, the 'paths (*Bahncurven*) of the corresponding infinitesimal transformation,' are

$$\frac{dp_1}{f_1}=\frac{dp_2}{f_2}=\dots=\frac{dp_n}{f_n}.$$

A *complete* system may then be characterized by saying that the lines of states are *fixed* in the corresponding space S_n, like the lines of flow of an incompressible fluid in steady motion. A copy of the system, or rather its representative point, placed on one of these lines remains on

it, moving along it in a determined sense. (For particulars of physical application of these concepts, see the author's paper in Ostwald's *Annalen d. Naturphilosophie*, Vol. II. pp. 201-254.)

Note 2 (to page 12). Systems obeying partial differential equations, as for instance that of Fourier,

$$\frac{\partial\theta}{\partial t}=a^2\nabla^2\theta=a^2\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right),$$

adduced in the text, may be considered as systems of infinite degrees of freedom. The instantaneous state of such a system implies an infinite number of data p_i, or say $p=p(x, y, z)$, given as a function of x, y, z for every point of a portion of space coextensive with the system, as for example the instantaneous temperature for every point of a cooling body of finite dimensions, in which case the system will have ∞^3 degrees of freedom. Instead of one we may have also two or more functions of x, y, z, defining the instantaneous state, as for example two vectors, amounting to six scalars, for an electromagnetic system (field), the differential equations being in this case those of Maxwell,

$$\frac{\partial \mathbf{E}}{\partial t}=c\,.\,\mathrm{curl}\,\mathbf{M},\quad \frac{\partial \mathbf{M}}{\partial t}=-c\,.\,\mathrm{curl}\,\mathbf{E}.$$

Here, as in the above example, the right-hand sides do not contain the time explicitly, but depend only on the space-distribution of magnitudes referring to the instantaneous state. If such be the differential equations and if also the limit or surface-conditions do not contain the variable t explicitly, the system of infinite degrees of freedom will be a *complete* or *undisturbed* one, in the sense of the word adopted throughout the chapter. Thus a heat-conducting sphere, of finite radius R, obeying in its interior Fourier's equation and whose surface is thermally isolated or radiates heat into free space, will be a complete system; for its boundary conditions, viz.

$$\frac{\partial\theta}{\partial r}=0$$

or

$$\frac{\partial\theta}{\partial t}=\mathrm{const.}\times(\theta-\mathrm{const.})$$

respectively, do not contain the time explicitly. But a sphere (like the earth), whose surface is kept at a generally variable temperature by means of external sources (like the sun), will be an incomplete system, unless we amplify it by taking in the 'sources' themselves.

CHAPTER II.

MAXWELLIAN EQUATIONS FOR MOVING MEDIA AND FRESNEL'S DRAGGING COEFFICIENT. LORENTZ'S EQUATIONS.

THE modern principle of relativity arose on the ground of Lorentz's electrodynamics and optics of moving bodies. Einstein's work, in fact, consisted mainly in deducing logically, on the basis of plausible and sufficiently general considerations, certain formulae of space and time transformation, which in Lorentz's theory had partly a purely mathematical meaning and partly the character of an hypothesis invented *ad hoc* ('local time' and the contraction hypothesis, respectively). In a word, Einstein has given a plausible support to, and a different interpretation of, what appeared already in the theory of the great Dutch physicist. In its turn, the theory of Lorentz, based on the macroscopic treatment of a crowd of electrons (though later supported and made vital by physical evidence of an entirely different kind), was constructed by its author chiefly with the purpose of accounting for optical phenomena in moving bodies, which may be best grouped summarily under the head of Fresnel's 'dragging coefficient' and with which the equations of Maxwell and of Hertz-Heaviside have proved to be in complete disagreement.

Now, it seems to me that the best, most natural and most efficient way of propagating new ideas (if indeed there is such a thing arising in the collective mind of humanity) is to show their intimate connexion with older ones, and the more so when the new ideas have the reputation, widespread but partly unjustified in our case, of being of a very revolutionary character. It will be advisable, therefore, before entering upon our proper subject, to turn back to Lorentz and Maxwell. In doing so, I must warn the reader at the outset that the new Relativity, though grown on electromagnetic soil, does not—in spite of a current opinion—require us at all to adopt an electro-

magnetic view of all natural phenomena. Nor does it force upon us a purely mechanistic view, which till recently held the field, before the pan-electric tendencies arose. Modern Relativity is broader than this: it subordinates mechanical, electromagnetic and other images to a much wider Principle which is colourless, as it were.

Thus, the reason of returning here to Maxwell is, in the first place, of an historical (and partly didactic) character. But we have yet another reason for dwelling in the present chapter upon the great inheritance left to Science by Clerk Maxwell. It is widely known that but a few things of the old system of physics have remained untouched by the modern principle of relativity, though the changes required are generally but very slight. In fact, almost nothing of the old structure has been spared by the new theory of relativity; but Maxwell's fundamental equations, namely those known as his equations for '*stationary*' media, have been spared. More than this: not only have they been preserved entirely in their original form, without the slightest modification of any order of magnitude whatever, but they form one and the best secured of the actual possessions of the new theory, the largest and brightest patch of colour, as it were, on the vast and as yet mostly colourless canvas contained within the frame of the new Principle. Moreover, a peculiar union or combination of the electric and magnetic vectors which appear in Maxwell's equations of the electromagnetic field became the standard and prototype (not as regards physical meaning, but mathematical transformational properties) of a very important class of entities admitted by the new theory (the so-called 'world-six-vectors' or 'physical bivectors').

So much to justify the insertion of the following topics of the present chapter.

Maxwell's fundamental laws of the electromagnetic field in a 'fixed' or 'stationary' non-conducting dielectric medium * may be expressed in integral form as follows:

I. Electric displacement-current through any surface σ bounded by the circuit $s = c \times$ line integral of magnetic force $\mathbf{M}$ round s.

II. Magnetic current through $\sigma = -c \times$ line integral of electric force $\mathbf{E}$ round s,

* Practically, fixed with respect to the earth, or, if not, then with respect to a definite system of reference S, to be ascertained on further examination.

i.e. in mathematical symbols:

$$\frac{d}{dt}\int(\mathfrak{E}\mathbf{n})\,d\sigma = c\int_{(s)}(\mathbf{M}\,d\mathbf{s}), \qquad \text{I.}$$

$$\frac{d}{dt}\int(\mathfrak{M}\mathbf{n})\,d\sigma = -c\int_{(s)}(\mathbf{E}\,d\mathbf{s}), \qquad \text{II.}$$

where $\mathfrak{E}$, $\mathfrak{M}$ denote the dielectric displacement or polarization and the magnetic induction respectively, c a scalar constant, the velocity of light in vacuum, $\mathbf{n}$ a unit vector normal to σ, the sense of the integration round s, of which $d\mathbf{s}$ is a vectorial element, being clockwise for a spectator looking along $\mathbf{n}$ (see Fig. 1). Here, as throughout

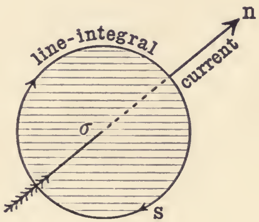


FIG. 1.

the volume, $(\mathfrak{E}\mathbf{n})$, etc., generally $(\mathbf{AB})$, in round parentheses, denote the *scalar product* of a pair of vectors:

$$(\mathbf{AB}) = AB\cos(\mathbf{A}, \mathbf{B}),$$

A, B being the sizes or absolute values of the vectors $\mathbf{A}$, $\mathbf{B}$.* Thus, the surface element $d\sigma$ being considered as an ordinary scalar, the surface integral $\int(\mathfrak{E}\mathbf{n})d\sigma$ stands for the total number of Faraday tubes (unit tubes) crossing σ, and the surface integral in II. has a similar meaning with respect to the tubes of magnetic induction.

* If it were only for purely vectorial algebra and analysis, we could write, after Heaviside, for the scalar product simply $\mathbf{AB}$. But since we shall have to recur in the sequel to Hamilton's quaternionic calculus, we reserve $\mathbf{AB}$ for the *full* quaternionic product, and write therefore $(\mathbf{AB})$ for the scalar product, *i.e.* for the negative scalar part of the Hamiltonian product, and $\mathrm{V}\mathbf{AB}$ for the vector product, thus

$$\begin{aligned}\mathbf{AB} &= \mathrm{S.}\,\mathbf{AB} + \mathrm{V.}\,\mathbf{AB}\\ &= -(\mathbf{AB}) + \mathrm{V}\mathbf{AB}.\end{aligned}$$

Remembering the definition of 'curl' by means of the line integral, we may write I. and II. at once in differential form,

$$\left.\begin{aligned}\frac{\partial\mathfrak{E}}{\partial t}&=c\,.\,\mathrm{curl}\,\mathbf{M},\\ \frac{\partial\mathfrak{M}}{\partial t}&=-c\,.\,\mathrm{curl}\,\mathbf{E},\end{aligned}\right\}\tag{1}$$

or, in Cartesian expansion,

$$\left.\begin{aligned}\frac{1}{c}\frac{\partial\mathfrak{E}_1}{\partial t}&=\frac{\partial M_3}{\partial y}-\frac{\partial M_2}{\partial z}, \quad \text{etc.},\\ \frac{1}{c}\frac{\partial\mathfrak{M}_1}{\partial t}&=\frac{\partial E_2}{\partial z}-\frac{\partial E_3}{\partial y}, \quad \text{etc.}\end{aligned}\right\}\tag{1a}$$

Every point or surface element of σ being *fixed*, relatively to the system of coordinates x, y, z, round ∂'s have been written on the left hand to express partial differentiations with respect to t, *i.e.* *local* time-rates of change of the corresponding vectors.

(1*a*) or (1) is the *Hertz-Heaviside form* of Maxwell's differential equations, although, if I am not mistaken, Maxwell himself on one occasion employed this form. At any rate, the Hertz-Heaviside equations for a stationary medium differ only formally from the equations of Maxwell as given in his monumental 'Treatise' and in several papers; the auxiliary potentials being easily eliminated.

As regards the relations obtaining between $\mathfrak{E}$, $\mathfrak{M}$ and $\mathbf{E}$, $\mathbf{M}$ respectively, it will be enough to remember here that the first pair of vectors are linear functions of the second, say,

$$\mathfrak{E}=K\mathbf{E} \quad \text{and} \quad \mathfrak{M}=\mu\mathbf{M},\tag{2}$$

where K, μ are in the general case, of crystalline bodies, symmetrical or self-conjugate linear vector operators, which in the simplest case of an isotropic medium degenerate into ordinary scalar coefficients, the dielectric 'constant' or the *permittivity*, and the magnetic permeability or the *inductivity*,—to adopt Heaviside's nomenclature.*

Notice that, using the relations (2), K and μ being supposed given, we have in (1) two vectorial equations of the first order for two vectors, so that if the 'initial' state, say $\mathbf{E}_0$, $\mathbf{M}_0$, and eventually the limit-conditions, be given, the whole history of the field, past

* As yet we have no need to touch upon the subject of *conducting* media.

and future, is uniquely determined,—though in most cases the mathematician may have the greatest difficulties in finding it out. The electromagnetic field, as far as it obeys these equations, is at any rate a complete system in the sense of the word previously explained. It will be noticed later that the fundamental equations of the electron theory do not possess this simple property.

From I., II. we see immediately that the total current, electric or magnetic, through all possible surfaces σ bounded by one and the same circuit (s), has the same value. Taking therefore a pair of such surfaces σ_1, σ_2, which together form a surface (σ), enclosing completely a certain portion τ of the medium, and inverting one of the normals of the component surfaces (Fig. 2), so that the

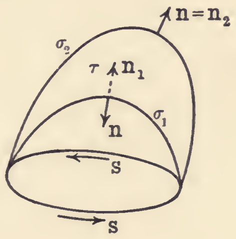


FIG. 2.

normal $\mathbf{n}$ is directed everywhere outwards (or everywhere inwards) with respect to the enclosed space, we see that, for any closed surface (σ),

$$\int_{(\sigma)} (\mathfrak{E}\mathbf{n})\,d\sigma, \quad \int_{(\sigma)} (\mathfrak{M}\mathbf{n})\,d\sigma = \text{const. in time,}$$

the second constant being everywhere equal to *zero*, by experience. In other words, the total electric charge enclosed by (σ) does not vary in time, its magnetic analogue being constantly non-existent. The same property being valid for any volume τ, and remembering that 'div' or divergence is defined as the surface integral of a vector per unit of enclosed volume, we may write also, in differential form,

$$\operatorname{div} \mathfrak{E} = \rho = \text{const.},$$

$$\operatorname{div} \mathfrak{M} = \text{const.} = 0\,;$$

ρ is the volume density of (true) electricity. The second property is commonly expressed by saying that the tubes of magnetic induction are always closed, or that $\mathfrak{M}$ has a purely *solenoidal* distribution. The invariability of both divergences may be seen with equal ease from (1), remembering that the operations div and $\partial/\partial t$ are commutative, while div curl = 0, identically.

Thus, the full system of Maxwell's equations for a stationary dielectric, which we will put here together for future reference, is

$$\left.\begin{aligned} \frac{\partial \mathfrak{E}}{\partial t} &= c \,.\, \mathrm{curl}\, \mathbf{M} \\ \frac{\partial \mathfrak{M}}{\partial t} &= -c \,.\, \mathrm{curl}\, \mathbf{E}; \quad \mathrm{div}\, \mathfrak{M} = 0 \\ \mathfrak{E} &= K\mathbf{E}; \quad \mathfrak{M} = \mu \mathbf{M}, \end{aligned}\right\} \tag{3}$$

the equation

$$\rho = \mathrm{div}\, \mathfrak{E}$$

being here considered as the definition of the density ρ of electric charge. Notice, in passing, that the 'electric charges' have been driven to the background by the Maxwellian theory (especially as propagated by Hertz, Heaviside and Emil Cohn), as rather secondary derivate entities, but to return later with increased vigour and to reacquire their dominant position, viz. as fundamental elements of the electron theory.

We shall not stop here to consider the general Maxwellian expressions of energy, ponderomotive force and of the corresponding stress.

In *vacuo*, and practically also in air under ordinary conditions,

$$K = \mu = 1,$$

so that Maxwell's equations (3) become

$$\left.\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= c \,.\, \mathrm{curl}\, \mathbf{M} \\ \frac{\partial \mathbf{M}}{\partial t} &= -c \,.\, \mathrm{curl}\, \mathbf{E} \\ \mathrm{div}\, \mathbf{M} &= 0, \end{aligned}\right\} \tag{4}$$

to which in the present case may be added also

$$\mathrm{div}\, \mathbf{E} = 0 \tag{4$_1$}$$

expressing the absence of electric charge. Notice in passing that these equations are *not* invariant with respect to the Newtonian

transformation. The transformation which does preserve their form is of a different kind, as will be seen later.

The independent variable t appearing in Maxwell's equations (4) for empty space may be taken, provisionally at least, as far as experience goes, to be the ordinary or the kinetic time. And as regards the (or a) space-framework, with respect to which they are intended to be rigorously valid, let us call it once and for ever the system S, whatever it may be. If the reader wants to fix his ideas he may think of S as the 'fixed-stars' system; but as yet we cannot and need not discuss this point thoroughly, being forced by the very nature of the question to postpone it to a later chapter. At first sight it might seem that (4) are wholly independent of a space-frame of reference; for the curls and div's can be, and primarily are, defined in terms of line integrals and surface integrals respectively, and thus depend only upon the distributional peculiarities of the respective vector fields. But this means only that the equations in question are independent of the choice of axes (x, y, z) within S, the only condition being that they must be immovable relatively to S; in other words, curl $\mathbf{E}$, curl $\mathbf{M}$ are vectors as good as $\mathbf{E}$, $\mathbf{M}$ themselves * and div $\mathbf{E}$, div $\mathbf{M}$ are true scalars like a volume, for instance. Notice, however, that, on the left hand of the equations, $\partial/\partial t$ is to be the *local* time rate of change of $\mathbf{E}$ or $\mathbf{M}$, *i.e.* the variation in a point P kept fixed. Now, this would be altogether meaningless if it is not explained with respect to what frame the point P is to be fixed. It would not help us very much if somebody told us that P is to be a fixed point of the field or of a Faraday tube; for we have no means of identifying such a point. The truth of what has just been said may be seen even more immediately from the integral form of Maxwell's equations, I. and II., where for the present case $\mathfrak{E}$, $\mathfrak{M}$ are to be identified with $\mathbf{E}$, $\mathbf{M}$; for the circuit (s) is to be kept 'fixed,' *i.e.* fixed with respect to something.† Therefore we necessarily want a frame of reference, and call it S.

* The distinction of what are called *axial* and *polar* vectors does not concern us here.

† In the more general case of a ponderable medium, say in a piece of glass, the circuit (s) is, of course, to be fixed in the glass; but this would not be enough: the whole piece of glass, as will be explained presently, must not move in an arbitrary manner relatively to some external frame or other, if the laws I., II. are to be valid, whether the observer does or does not share its motion.

To see the property of the scalar constant c, eliminate, in the usual way, **E** or **M**, employing their solenoidal properties; then

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi, \tag{5}$$

where ϕ means **E** or **M**, or any one of their Cartesian components $E_1, \dots M_3$; hence, in the case of plane waves, for example,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\frac{\partial^2\phi}{\partial x^2}$$

and

$$\phi=f(x\pm ct),$$

f being an arbitrary function of the linear argument. Thus c, in round figures $3 \cdot 10^{10}$ cm. sec.$^{-1}$, is the *velocity of propagation* in empty space, relatively to S, of transversal electromagnetic waves or disturbances, their transversality being an immediate consequence of the solenoidal conditions, which, in the present case, reduce to $\partial E_1/\partial x=0$, $\partial M_1/\partial x=0$. Henceforth c will be referred to shortly as 'the velocity of light,' and sometimes as the 'critical' velocity.

What is properly called a **wave** is a non-stationary surface of discontinuity of **E**, **M** themselves or of their derivatives, which is individually recognizable as such and can be watched when moving about. It is the velocity of motion of such a wave, normal to itself, which is properly called the **velocity of propagation**, as distinguished from the phase-velocity of a continuous train of disturbances. Now it may be easily shown that c is precisely the value of this true velocity of propagation for any form of the wave, plane or not, the property belonging to every surface element of the wave, considered separately. (See **Note 1** at the end of the chapter.)

Notice that this property is quite independent of the direction of the wave normal, *i.e.* of its orientation with respect to any axes drawn in S. In other words:

> Maxwell's equations imply *isotropic* as well as uniform * propagation in empty space *relatively* to S, *i.e.* to that system in which they are valid. There are no privileged places or directions for the electromagnetic disturbances.

Thus a continuous train of spherical waves, with centre O, will remain spherical for ever, which may be seen also from (5). For a

* By 'uniform' we mean homogeneous or constant in space and invariable in time, c being constant with respect to both.

particular integral of that equation, adaptable to any initial state $\phi_0 = \frac{1}{r} f(r)$, is $\phi = \frac{1}{r} f(r \pm ct)$, r being the scalar distance measured from O. Again—which is more satisfactory—if σ be at any instant a spherical surface of transversal discontinuity or a proper electromagnetic wave, then, expanding (or shrinking) with time, it will remain spherical for ever, with centre O coinciding always with that of the original σ, *fixed once and for ever with respect to the frame* S,—quite independently of whether and how the material source was moving at the instant when it originated that wave. Thus a 'point-source' (and notice that a physical source of any shape or finite dimensions may be regarded as such, provided we go away from it far enough) producing a solitary disturbance, say a flash of light, at the instant t_0, will originate a wave which always will be spherical of radius

$$R = c(t - t_0),$$

having its centre where the source was at the instant t_0, no matter whither it went afterwards or whence it came, or how swiftly it flashed through that place.

We shall have to return to this argument, of capital importance, more than once; but meanwhile we must leave it.

As has been already remarked, Maxwell's equations for '*stationary*' dielectrics, *i.e.* I. and II. with their supplements as given together with their differential form under (3), have not only survived the general massacre, but have very substantially enriched the new theory. In fact, both the most particular and simple equations (4) for the vacuum and the more general ones, (3), for ponderable media have been incorporated into the possessions of modern Relativity, the former in a quite easy way by Einstein (1905), and the latter in a less easy and very ingenious way by Minkowski (1907). On the other hand, it is needless to tell here again about the wide field of experience covered by these equations and about their numerous and successful applications in proper Electromagnetism, to say nothing about the electromagnetic theory of light which soon after its creation proved to be much superior to the elastic theory.

Serious difficulties arose only in connexion with the electrodynamics, and more especially with the optics of *moving* media, a long time before the dates just quoted.

There are two different sets of what are commonly called Maxwellian equations for moving media: 1° a system of equations which may be gathered together from different chapters of Maxwell's 'Treatise,' and which we shall call shortly *the equations of Maxwell*, though it can be reasonably doubted whether Maxwell himself would consent to attribute to them general validity, especially with the inclusion of optics; and 2° a system of equations which Hertz obtained by a certain, apparently the most obvious, extension of the meaning of the form I., II., and which Heaviside, independently, constructed by introducing into Maxwell's equations a supplementary term dictated by reasons of electro-magnetic symmetry; these are widely known as the *Hertz-Heaviside equations* for moving bodies.

We shall use for 1° and 2° the abbreviations (Mx), (HH). Neither has been able to stand the test of experience. Though contrary to the historical order, it will be more instructive to consider first the latter and then the former system of equations.

Let us return to the semi-integral form of electromagnetic laws I. and II., given, in words and symbols, on pp. 22-23. These are valid for a ponderable dielectric medium or body, stationary with respect to our frame S, and for any surface σ which, together with its bounding circuit s, is fixed in the body. Thus the surface σ, through which the 'current' is to be taken, is itself fixed in S. Now, what Hertz did in order to obtain the required extension, was simply to suppose that I. and II. are still valid for a body, rigid or deformable, moving with respect to S in any arbitrary manner, provided that the currents on the left-hand side of these equations are taken through a surface composed always of the same particles of the body, or—to put it shortly—through an *individual* σ, together with its s. This gives for the current per unit area of σ, instead of the local time-rate of change $\partial\mathfrak{E}/\partial t$, if $\mathbf{v}$ be the velocity of a particle relatively to S,

$$\frac{\partial\mathfrak{E}}{\partial t}+\mathbf{v}\operatorname{div}\mathfrak{E}+\operatorname{curl}\mathrm{V}\mathfrak{E}\mathbf{v}, \tag{6}$$

and a similar expression for the magnetic current,* while the right-hand sides of I., II., containing only the instantaneous values of line integrals, remain obviously unaffected by the Hertzian requirement. The distribution of $\mathfrak{M}$ being supposed solenoidal, as

* See **Note 2.**

previously, the second term in the above expression is absent in the magnetic current. Thus, transferring the curl-terms of the currents to the right-hand sides, we obtain the required equations

$$\left.\begin{aligned} \frac{\partial \mathfrak{E}}{\partial t} + \rho \mathbf{v} &= c \,.\, \mathrm{curl}\left(\mathbf{M} - \frac{1}{c}\mathrm{V}\mathfrak{E}\mathbf{v}\right) \\ \frac{\partial \mathfrak{M}}{\partial t} &= -c \,.\, \mathrm{curl}\left(\mathbf{E} - \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}\right). \end{aligned}\right\} \qquad \text{(HH)}$$

Heaviside calls $\mathrm{V}\mathfrak{E}\mathbf{v}/c$ the 'motional magnetic force' and $\mathrm{V}\mathbf{v}\mathfrak{M}/c$ the 'motional electric force,' considering them as a kind of impressed forces.

In what we have called Maxwell's equations, the former of these 'motional forces' and the convection current $\rho\mathbf{v}$ are absent; otherwise they are as (HH); thus

$$\left.\begin{aligned} \frac{\partial \mathfrak{E}}{\partial t} &= c \,.\, \mathrm{curl}\,\mathbf{M} \\ \frac{\partial \mathfrak{M}}{\partial t} &= -c \,.\, \mathrm{curl}\left(\mathbf{E} - \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}\right). \end{aligned}\right\} \qquad \text{(Mx)}$$

The connexions between $\mathfrak{E}$, $\mathfrak{M}$ and $\mathbf{E}$, $\mathbf{M}$ are as in (3), except that K, μ may undergo continuous variations due to the strain of the material medium. Also, $\mathrm{div}\,\mathfrak{M} = 0$, as in (3). Notice, in passing, that the first of (HH) gives

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0$$

or

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\mathbf{v} = 0,$$

where $d\rho/dt = \partial\rho/\partial t + (\mathbf{v}\nabla)\rho$ is the variation at an individual point of the body. Now, $\mathrm{div}\,\mathbf{v}$ being the cubic dilatation, per unit time and per unit volume, the last equation may at once be written

$$\frac{d}{dt}(\rho\, d\tau) = 0,$$

where $d\tau$ is an *individual* volume-element of the material medium, *i.e.* an element composed always of the same particles. Thus the charge $\rho d\tau$ of any such element remains invariable, being attached to it once and for ever. The charge, being preserved in quantity, moves

with the body. In this respect it behaves like the mass, according to classical mechanics. As regards the equations (Mx), they must be considered as referring to the particular case of an uncharged body; Maxwell happened not to consider explicitly charges in motion; otherwise he would doubtless have brought in the term $\rho\mathbf{v}$.

Now, both of these systems of equations, (Mx) as well as (HH), are in full disagreement with experience, especially with optical experience, terrestrial and astronomical, *i.e.* with experiments on the propagation of electromagnetic waves (light) in bodies moving relatively to the observer, and also in bodies moving with the observer and with his apparatus relatively to the source, say relatively to a star.

The equations in question have also been manifestly contradicted by electromagnetic experiments properly so called, viz. those of H. A. Wilson and of Roentgen and Eichenwald;* but it will be enough to consider here only the difficulties met with on optical ground, the other deviations being of essentially the same character, while the optical examples, quite conclusive by themselves, seem to be very instructive.

Let me explain to you fully what this disaccordance consists in.

To take the simplest case possible, let the material medium or

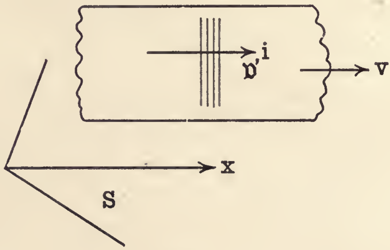


Fig 3.

body move as a whole with uniform translational velocity $\mathbf{v}$ with respect to S, and let plane waves of light be propagated in it along the positive direction of $\mathbf{v}$ (Fig. 3). If the unit-vector $\mathbf{i}$ be the wave normal, concurrent with the propagation, then $\mathbf{v}=v\mathbf{i}$. Let $\mathfrak{b}'$ be the scalar velocity of propagation of the waves, when the material medium

* H. A. Wilson, *Phil. Trans.*, *A.* Vol. CCIV. p. 121; 1910.—W. C. Roentgen, *Berl. Sitzber.*, 1885; *Wiedem. Ann.*, Vol. XXXV. 1888, and Vol. XL. 1890.—A. Eichenwald, *Ann. der Physik*, Vol. XI. 1903.

is stationary in S, and $\mathfrak{v}$ their velocity of propagation, as judged from the S-standpoint, when the medium is moving with its actual velocity. What is the relation between $\mathfrak{v}$ and $\mathfrak{v}'$, v? If we were concerned with waves of sound, instead of light waves, then $\mathfrak{v}$ would be simply the sum of $\mathfrak{v}'$ and of the whole v; the waves would be entirely dragged by the medium, say air or water, with its full velocity. But the case before us is different. Write, generally,

$$\mathfrak{v} = \mathfrak{v}' + \kappa v$$

or

$$\kappa = \frac{\mathfrak{v} - \mathfrak{v}'}{v};$$

then κ, whatever its value, will be what is called the **dragging coefficient**, indicating the fraction (if it happens not to be the whole) of the medium's velocity conferred upon the waves. What is, then, the dragging coefficient in the case of electromagnetic, and especially of luminous waves?

According to (HH) it is, obviously, equal to *unity*. To see this we have no need to integrate these differential equations,* but simply to remember Hertz's interpretation of the laws I., II., which furnished him with these equations (p. 30). For according to that interpretation, and extension, of I., II., the electromagnetic disturbances behave *relatively to the material medium* (generally in each of its elements, and in the present case, of rigid translation, throughout the whole medium) just as if it were stationary. Hence, on the ground of classical kinematics of course, the velocity of the medium is simply added to that of the waves, precisely as in the case of sound. Thus, $\kappa = 1$, according to (HH).

Let us now see what is the value of the dragging coefficient according to (Mx). Take the simplest case of an isotropic medium; then

$$\mathfrak{v}' = \frac{c}{\sqrt{K\mu}},$$

where, by the way, $\mu = 1$ for light waves. Measuring x along $\mathbf{i}$ in the system S, take **E**, **M**, and therefore also $\mathfrak{E}$, $\mathfrak{M}$, proportional to a function of the argument $x - \mathfrak{v}t$, so that $\mathfrak{v}$ will be the velocity of

* Though the reader, to satisfy himself, may do so. Proceeding similarly as in the case of (Mx), worked out in **Note 3** at the end of this chapter, he will soon find that $\mathfrak{v} = \mathfrak{v}' + v$.

propagation relatively to S, as above, and by a simple calculation (Note 3)

$$\mathfrak{b} = \sqrt{\mathfrak{b}'^2 + \tfrac{1}{4}v^2} + \tfrac{1}{2}v$$

or

$$\mathfrak{b} = \mathfrak{b}'(1 + \tfrac{1}{4}n^2\beta^2)^{\frac{1}{2}} + \tfrac{1}{2}v, \tag{7}$$

where $\beta = v/c$ and where $n = c/\mathfrak{b}'$ is the index of refraction of the medium. Now, in all actual experiments, by means of which the dragging of light can be determined, β is a small fraction, viz. 10^{-4} in the case of Airy's astronomical, and much smaller in that of Fizeau's terrestrial experiment, both to be considered later. Therefore terms of the order of β^4 can certainly be rejected, so that

$$\mathfrak{b} = \mathfrak{b}' + \frac{1}{2}v + \frac{1}{8}nc\beta^2$$

and

$$\kappa = \frac{1}{2}\left(1 + \frac{n}{8}\beta\right); \tag{7a}$$

but here even the β-term may be safely omitted, so that finally

$$\kappa \doteqdot \tfrac{1}{2}.*$$

Thus, we have for the dragging coefficient according to (HH) and (Mx), respectively,

$$\kappa = 1, \tag{HH}$$

$$\kappa \doteqdot \tfrac{1}{2}. \tag{Mx}$$

Now, both of these are radically wrong, the true one, *i.e.* that showing excellent agreement with experiment, being Fresnel's widely known dragging coefficient (*coéfficient d'entraînement*)

...of the sines
...gle of incident (with perpendicular)
...gle of refraction (" ")

$$\kappa = 1 - \frac{1}{n^2}, \tag{Frsnl}$$

sin α / sin β

where n is the index of refraction. It is, for more than one reason, worth our while to dwell here upon the interesting history of Fresnel's coefficient.

The phenomenon of stellar *aberration*, discovered by Bradley in 1728, found its immediate explanation when the assumption was made that the light-waves do not share in the earth's orbital motion;

Bradley

* This result was obtained by J. J. Thomson. See Heaviside's *Electromagnetic Theory*, Vol. III, § 471 *et seq.*, where some interesting remarks regarding this and allied subjects may be found.

and, consequently, in the motion of the tube of the telescope (if filled with air or empty). In fact, making this assumption, the aberrational formula

$$\frac{v}{c}=\frac{\sin\phi}{\sin\theta} \quad \text{(see Fig. 4)} \tag{8}$$

and, for $\theta=\pi/2$, – 90°

$$\frac{v}{c}=\sin\phi \doteqdot \tan\phi, \tag{8a}$$

is easily obtained by using the widely known analogy of a ship in motion pierced by a shot fired from a gun on the shore.

Formula (8) gave, from Bradley's observations ($\phi=20''\cdot 44$) and from the known velocity v of the earth's motion (30 kilom. per second), a value for c, the velocity of propagation of light, which agreed very

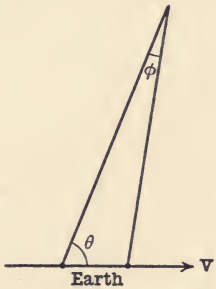


FIG. 4.

closely with that obtained by Römer in 1676 from observations of the eclipse of Jupiter's satellites. Thus (8) was verified. To state the bare facts, it would have been enough to say simply that the tube of the telescope, or the air contained in it, does not carry with it the light coming from the star, whatever it may consist in (corpuscles or waves). But to make the statement more tangible, it has been said that the 'corpuscles' or the 'aether,' respectively, do not share in the telescope's motion. Whereas aberration was explained by its discoverer in terms of the corpuscular theory (each corpuscle of light corresponding then most immediately to the shot in the above analogy), it was Young who first showed (1804) how it may be explained on the wave-theory of light and on the hypothesis that the aether 'pervades the substance of all material bodies with little or no resistance, as freely perhaps as the wind passes through a grove of

trees.'* This picturesque analogy fitted altogether the case of air, which behaves very nearly like a vacuum, but not glass or water, for which the 'grove of trees' had to be replaced by a rather dense thicket. But at any rate the above words of Young hit very near the truth.

To put it shortly, in the case of air the dragging is *nil*, or nearly so, $\kappa \doteqdot 0$.

But the case is different for optically denser media, having, for light of a given frequency, an index of refraction n, sensibly different from unity. For if κ were nil also for such media, we should have to replace c in (8) by the smaller velocity of propagation c/n, so that the angle of aberration would be different for optically different media, whereas it has been proved experimentally to be just the same as in the case of air. More generally, Arago concluded from his experiments on the light of stars that the earth's motion has no sensible influence on the refraction (and reflection) of the rays emitted by these light-sources, *i.e.* that the rays coming from a star behave, say, in the case of a prism or a slab of glass, precisely as they would if the star were situated at the point in which it appears to us in consequence of ordinary Bradleyan (air-telescope) aberration, and the earth were at rest relatively to the star. Arago himself tried to explain this result of his experiments on the corpuscular theory, and on the supplementary hypothesis that the sources of light impress upon the corpuscles an infinity of different velocities, and that out of these none but those endowed with a certain velocity ($\pm \cdot 01\,\%$) have the power of exciting our organ of sight. But this strange hypothesis entangled him in a maze of difficulties, and the whole theory, not free from other difficulties, does not seem to have satisfied its author. At any rate, Arago proposed to Fresnel to investigate whether the above result of his observations could not be more easily reconciled with the wave theory of light.

1818 It was in answer to this invitation that Fresnel wrote in 1818 his celebrated letter to Arago 'on the influence of the earth's motion upon certain optical phenomena,'† in which he gives a beautiful

* *Phil. Trans.*, 1804, p. 12,—as quoted by Whittaker in *A History of the Theories of Aether and Electricity*, p. 115; London, 1910.

† 'Lettre d'Augustin Fresnel à François Arago, sur l'influence du mouvement terrestre dans quelques phénomènes d'optique,' *Annales de chim. et de phys.*, Vol. IX. p. 57, cahier de septembre, 1818; reprinted in Fresnel's *Œuvres complètes*, Vol. II., Paris, 1868; No. XLIX. pp. 627-636.

solution of the problem, and which has since become one of the most solid supports of modern inquiry into the optics of moving media. Here appears for the first time his 'coéfficient d'entraînement,' already mentioned above. Fresnel based the theory of aberration, and associated matters, on the following hypothesis, which turned out to be a very happy guess indeed:

> Fresnel supposed that the *excess*, and only the excess, of the aether contained in any ponderable body over that in an equal volume of free space *is carried along with the full velocity*, v, *of the body*; while the rest of the aether within the space occupied by the body, like the whole of the free aether outside, is stationary,—with respect to the fixed stars, of course.

This amounts * to supposing that the velocity of propagation of the light-waves is augmented only by the velocity of the '*centre of gravity*' (centre of mass) of the whole mass of the aether contained in the body. This velocity will, generally, be but a fraction of v. Call it κv; then κ will be what has above been called the dragging coefficient. Let ρ_0 be the density of the aether outside the body, and ρ its density within the body; then, by Fresnel's hypothesis,

$$(\rho - \rho_0)v = \rho \, . \, \kappa v$$

or

$$\kappa = 1 - \rho_0/\rho.$$

Now, e being the coefficient of elasticity of the aether within the body, and e_0 that of the free aether, the body's refractive index n is given by

$$n^2 = \frac{e_0}{\rho_0} \bigg/ \frac{e}{\rho}.$$

But Fresnel's aether has throughout the same elasticity, within ponderable bodies and interplanetary space, so that $e = e_0$ and $n^2 = \rho/\rho_0$.

Thus we obtain Fresnel's celebrated formula for the dragging coefficient:

$$\kappa = 1 - \frac{1}{n^2}. \qquad \text{(Frsnl)}$$

Notice that considering the excess of the aether, *i.e.* $\rho - \rho_0$ per unit volume, as a permanent part of material bodies, it can be said simply that the *aether proper* is not moved at all, that it is entirely

* See the letter in question, p. 631 of reprint in Vol. II. of *Œuvres complètes*.

uninfluenced by the moving bodies. Fresnel's theory is therefore usually alluded to as the theory of a *fixed aether*. Implicitly, this aether of Fresnel is supposed to be fixed relatively to the stars, or at least to those stars which have been concerned in the aberrational observations.

For a vacuum, or air, $n=1$ and $\kappa=0$. Thus, first of all, Fresnel's theory is in perfect agreement with Bradley's observations. For other media $n>1$ and $0<\kappa<1$, or the dragging is *partial*, and increases with the optical density of the medium.

By means of his dragging coefficient Fresnel treated fully the problem of refraction in a prism, showing that it must be sensibly* uninfluenced by the earth's motion, in agreement with Arago's observations. This problem, in fact, was the chief object of the letter quoted.

To close his admirable letter, Fresnel gives an application of his theory to an experiment, suggested previously, in 1766, by Boscovich,† consisting in the observation of the phenomenon of aberration with a telescope filled with water,—commonly called 'Airy's experiment.' Fresnel infers from his formula for κ, by simple and most elegant reasoning, that if observations were made with such a telescope, the aberration would be unaffected by the presence of the water. This result was verified, for the first time, by Sir G. B. Airy in 1871, in the observatory of Greenwich. His observations on γ Draconis, during 1871-1872, proved indeed that the presence of water, in place of air, has no sensible, *i.e.* no first-order (v/c) influence on the aberration.

* *i.e.* as far as the *first* power of v/c goes.

† R. J. Boscovich (or Bošković), born in Ragusa 1711, died in Milan 1787. The principle of the water-telescope was first explained by Boscovich in a letter to Beccaria in 1766, and then fully developed in the second volume of his optical and astronomical papers, *Opera pertinentia ad opticam et astronomiam;* Basani, 1785, Vol. III, opusculum III. pp. 248-314. An interesting account of the work (and life) of Boscovich is given by G. V. Schiaparelli in a manuscript, *Sull' attività del Bošković quale astronomo in Milano*, edited recently by Dr. V. Varićak (Agram, South Slavic Acad. of Sc., 190; 1912). In connexion with the subject of our Chap. I., the reader may also be warmly recommended to consult another paper of Boscovich, edited by Dr. Varićak (*ibidem*, 190; 1912): *De motu absoluto, an possit a relativo distingui*, originally a supplement of Boscovich to *Philosophiae recentioris a Benedicto Stay versibus traditae*, Libri X, ; Vol. I. p. 350; Rome, 1755. This paper, which is missing even in Duhem's bibliography of the subject (*Le mouvement absolu et le mouvement relatif*, 1909), contains many remarkably clear and radical ideas regarding the relativity of space, time and motion.

For both of these pamphlets I am indebted personally to Dr. Varićak.

Though Fresnel's own reasoning, reprinted at the end of the present chapter (Note 4), exhausts the subject entirely, let us yet dwell upon it a moment.

If the aether behaved in optically denser bodies as in air, *i.e.* if there were no dragging at all, we should have, by the ship and shot analogy, instead of (8),

$$\frac{v}{c/n}=\frac{\sin\phi}{\sin\theta},$$

c/n being the velocity of propagation of light in water, or in any other medium filling the tube of the telescope. Then Airy's experiment would have given a positive result. But he obtained precisely the same ϕ as for air. This negative result suggested to him (at least as it is usually represented in text-books) the supposition that the 'water carries with it the aether' with only a certain part of its velocity, namely such that, in the above formula, we have to write $\bar{v}$ instead of v, where

$$\bar{v}=v/n,$$

so that

$$\frac{\sin\phi}{\sin\theta}=\frac{\bar{v}}{c/n}=\frac{v}{c},$$

as for air. In reality the process of compensation is not so simple as this; but in Airy's experiment the compensation—sensibly complete—is produced in a slightly different way. Considering a slab of water moving perpendicularly to its axis, and neglecting second-order terms (*i.e.* $v^2/c^2=10^{-8}$), you will easily obtain *

$$\frac{\sin\phi}{\sin\theta}=\frac{(v-\bar{v})c}{c^2/n^2}=(1-\kappa)\frac{vn^2}{c}, \tag{9}$$

where, $v-\bar{v}$ being the relative velocity of the aether and telescope, $\kappa=\bar{v}/v$ has been written for the dragging coefficient, as yet supposed to be unknown. Hence, to account for Airy's negative result, *i.e.* to make (9) identical with (8), we have to write $(1-\kappa)n^2=1$, or

$$\kappa=1-\frac{1}{n^2},$$

as in Fresnel's formula.

* See, if necessary, for instance N. R. Campbell's *Modern Electrical Theory*, Cambridge, 1907; pp. 293-294 (but interchange the dashes at *P*, *C*, *O*, *Q* in his Figure 28, which are placed the wrong way; correct also some dashes on p. 294 and read at the bottom of the page 'presence' instead of 'pressure.' As regards Fizeau's experiment, amend the shocking anachronism on p. 295: 'Fizeau tried'—1851—'to test the correctness of Airy's hypothesis'—1871).

Thus, Airy's negative result is perfectly accounted for by Fresnel's dragging coefficient, terms of the order of 10^{-8} being, of course, beyond the possibility of observation.

But Fresnel's formula found also, twenty years earlier, an im-
1851 mediate verification in Fizeau's optical interference-experiment with flowing water.* The arrangement of the apparatus which was used by Fizeau is seen at a glance from Fig. 5. Light from a narrow slit, S, after reflection from a plane parallel plate of glass, AA, is rendered parallel by a lens L and separated into two pencils by apertures in a screen EE placed in front of the tubes T_1, T_2 containing running water. The two pencils, after having traversed (towards the left hand) the respective columns of water, are focussed, by the lens B, upon a plane mirror Z, which interchanges their paths: the upper pencil returns towards L by the tube T_2, the lower by T_1. On

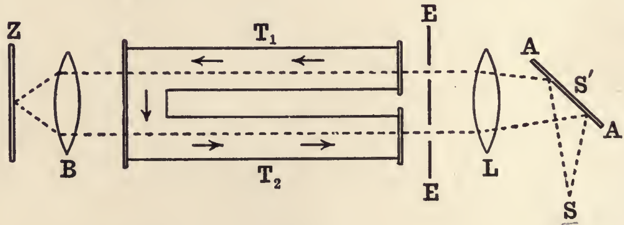


FIG. 5.

emerging finally from the water, both pencils are brought, by L, to a focus behind the plate AA, at S' (and partly also at S). Here a system of interference fringes is produced which can be observed and measured in the usual way. Thus, each pencil traverses both tubes, T_1 and T_2, *i.e.* the same thickness of flowing water, say l. Moreover, the (originally) upper pencil is travelling always with, the other against the current. If, therefore, v be the velocity of the water and κ the dragging coefficient, the difference in light-time for the two pencils will be given by

$$\Delta = l\left\{\frac{1}{c/n - \kappa v} - \frac{1}{c/n + \kappa v}\right\},$$

where n is the refractive index of water. Passing from stationary to flowing water, Fizeau observed a measurable displacement of the interference fringes, namely with $v = 700$ cm./sec.; and by reversing

* H. Fizeau, *Comptes rendus*, Vol. XXXIII., 1851; *Annales de Chimie*, Vol. LVII., 1859.

the direction of the current of water the displacement of the fringes could be doubled. From the observed displacement it is easy to find the difference of times Δ, and by equating it to the above expression of Δ to find the dragging coefficient κ in terms of l, n, v, which can be measured. The result of Fizeau's experiment was that κ is a fraction, sensibly less than unity. How much less, could not be ascertained with sufficient precision. Fizeau's experiment was therefore repeated in a form modified in several important points by Michelson and Morley * (1886), who found, for water (moving with the velocity of 800 cm. per second) at 18° C., and for sodium light,

$$\kappa = 0{\cdot}434 \pm 0{\cdot}02, \tag{MM}$$

i.e. 'with a possible error of $\pm 0{\cdot}02$.'

Now, n being, in the case in question, equal to 1·3335, Fresnel's formula gives

$$\kappa = 1 - \frac{1}{n^2} = 0{\cdot}438, \tag{Frsnl}$$

a value agreeing very closely with Michelson and Morley's experimental result.

Thus, Fresnel's formula, deduced from what in our days may be deemed an assumption of naïve simplicity, proved to be in admirable conformity with experiment, like everything predicted by Fresnel in optics. His dragging coefficient has acquired a special importance in recent times, and every modern theory is proud to furnish his κ, which has become, in fact, one of the first requirements demanded from every theory of electrodynamics and optics of moving bodies which is being proposed. 'Agreeing with Fresnel' has become almost a synonym of 'agreeing with experience.'

Now Maxwell's and Hertz-Heaviside's equations for moving media, (Mx) and (HH), giving, as we have just seen, $\kappa \doteqdot \frac{1}{2}$ and $\kappa = 1$, or *half* and *full* drag, respectively, for any medium, be it as dense as water or glass or as rare as air, proved thereby to be in full disagreement with Fresnel, *i.e.* with experiment.

The first successful attempts to smooth out this discordance of (Mx) and (HH) from experiment, which—as has been mentioned—manifested itself also in the case of electromagnetic experiments properly so called, were made by H. A. Lorentz in 1892. The Loren

* Michelson and Morley, *American Journ. of Science*, Vol. XXXI. p. 377; 1886. See also A. A. Michelson's popular book, *Light Waves and their Uses*; Chicago 1907; p. 155.

theory proposed in a paper published in that year,* and which led with sufficient approximation to Fresnel's dragging coefficient, was then simplified and extended in 1895, in a paper † which has since become classical.

Stokes' moving aether (1845) leading to serious difficulties,‡ Lorentz decided in favour of Fresnel's immovable, stationary aether, as the all-pervading electromagnetic medium.

Thus, Lorentz's theory, presently known widely as the Electron Theory, is, first of all, based on the assumption of a **stationary**, isotropic and homogeneous aether. In calling it shortly 'stationary' (*ruhend*), Lorentz states expressly that to speak of the aether's 'absolute rest' would be pure nonsense, and that what he means is only that the several parts of the aether do not move relatively to one another (*Essay*, p. 4). In other words, Lorentz's aether is not deformed, it is subjected to no strain, and does not, consequently, execute any mechanical oscillations. And this being the case, it has, of course, no kind of elasticity, nor inertia or density. It is thus far less corporeal than Fresnel's aether. One fails to see what properties, in fact, it still has left to it, besides that of being a colourless seat (we cannot even say substratum) of the electromagnetic vectors **E**, **M**. And although Lorentz himself continues to tell us, in 1909,§ that he 'cannot but regard the ether as endowed with a certain degree of substantiality,' yet, for the use he ever made of the aether, he might as well have called it an empty theatre of **E**, **M**, and their performances, or a purely geometrical system of reference, stationary with regard to the (or at least to some) 'fixed' stars. This aether, having been deprived of many of its precious properties, was at any rate already so nearly non-substantial, that the first blow it had to sustain from modern research knocked it out of existence altogether,—as will be seen later. Still, substantial or not, for the theory of Lorentz we are now considering, it is *something*, namely its unique system of reference. So long, therefore, as it was thought that there is such an

* H. A. Lorentz, *La théorie électromagnétique de Maxwell et son application aux corps mouvants*; Leiden, E. J. Brill, 1892 (also in *Arch. néerl.*, Vol. XXV.).

† H. A. Lorentz, *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern*; Leiden, E. J. Brill, 1895. This paper will be shortly referred to as '*Essay*' (= *Versuch*).

‡ See **Note 5** at the end of this chapter.

§ Lorentz, *The Theory of Electrons*, etc., Lectures delivered in Columbia University, 1906; Leipzig, Teubner, 1909; p. 230.

unique system, Lorentz's all-pervading medium could continue its scanty existence.

For this *free* aether, *i.e.* where it is not contaminated by the presence of ponderable matter, Lorentz assumes the exact validity of *Maxwell's equations*, (4), *i.e.*

$$\frac{\partial \mathbf{E}}{\partial t} = c \, . \, \mathrm{curl}\, \mathbf{M}\, ; \quad \frac{\partial \mathbf{M}}{\partial t} = -c \, . \, \mathrm{curl}\, \mathbf{E}\, ; \quad \mathrm{div}\, \mathbf{M} = 0,$$

with $\rho = \mathrm{div}\, \mathbf{E} = 0$. (As to terminology, Lorentz calls the above $\mathbf{E}$ the *dielectric displacement*, and $\mathbf{M}$ the *magnetic force*.)

Then, to account for the optical and, more generally, electromagnetic phenomena in moving ponderable matter, he has recourse to *electro-atomism*, an hypothesis already employed (1882-1888) by Giese, Schuster, Arrhenius, Elster and Geitel, and others, and later also by Helmholtz (1893) in his famous electromagnetic theory of dispersion, and in various writings of Sir Joseph Larmor. According to Lorentz, matter *by itself* has no influence whatever on the electromagnetic phenomena: in this respect it behaves like the free aether. Only when and as far as matter is the seat of 'ions,' in Lorentz's, or electrons in modern terminology,* it modifies the electromagnetic field and its variations. In other words, Maxwell's equations, (4), are assumed to be strictly valid not only in the free aether, but also in all those portions of ponderable molecules in which there is no charge, *i.e.* wherever $\rho = 0$. And as to the question whether ponderable matter consists entirely of electrical particles (charges) or not, Lorentz leaves it an open question. If I may venture an opinion, it was very wise of him *not* to have had M. Abraham's ambition to construct a purely electromagnetic 'Weltbild,' as the Germans call it. (This remark will be understood better later on, when we shall see that, as far as we know, even the mass of the free electrons, such as the kathode ray- or β-particles, may not be of purely electromagnetic origin.) The part played in Lorentz's theory by matter itself consists only in keeping the electrons, or at least some of them, at or round certain places, say, restraining them from too wide excursions. Maxwell's equations, as written above for the free aether, are modified only where

$$\mathrm{div}\, \mathbf{E} \equiv \rho \neq 0,$$

* 'Electron' is due to Johnstone Stoney (1891). The distinction made now between 'ions' and 'electrons' does not concern us here; besides, it is generally known from a host of popular writings.

i.e. where there is, at the time being, some electric charge or electricity, and where, moreover, the electricity is moving.* The 'modification is the slightest imaginable,' to put it in Lorentz's own words (*Electron Theory*, p. 12). If **p** be the velocity of electricity at a point, relatively to the aether, *i.e.* relatively to that system of reference, S, in which the free-aether equations (4) are valid, then the left-hand member of the first of these equations, or the *displacement current*, is supplemented by the *convection current*, per unit area, *i.e.* by $\rho\mathbf{p}$, while the second and third equations remain unchanged.

Thus, Lorentz's differential equations, assumed to be valid exactly or *microscopically*† throughout the whole space, are

$$\left.\begin{aligned} &\frac{\partial \mathbf{E}}{\partial t} + \rho\mathbf{p} = c\,.\,\mathrm{curl}\,\mathbf{M}, \quad \text{where } \rho = \mathrm{div}\,\mathbf{E} \\ &\frac{\partial \mathbf{M}}{\partial t} = -c\,.\,\mathrm{curl}\,\mathbf{E}\,; \quad \mathrm{div}\,\mathbf{M} = 0. \end{aligned}\right\} \qquad (\mathrm{I.})$$

These have been since generally called **the fundamental equations of the electron theory**. They contain, of course, the equations for the free aether as a particular case, namely for $\rho = 0$.

An important supplement to the above system of equations consists in the formula for **the ponderomotive force** 'acting on the electrons and producing or modifying their motion,' which, guided by obvious analogies, Lorentz assumes to be, *per unit volume*,

$$\mathbf{P} = \rho\left[\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{pM}\right], \qquad (\mathrm{II.})$$

or, per unit charge,

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\,\mathrm{V}\mathbf{pM}. \qquad (10)$$

This 'force' is supposed to be exerted by the aether on electrons or matter containing electrons. *Vice versa*, as Lorentz states it expressly, matter, whether containing electrons or not, exerts no action at all on the aether,—since the aether has already been supposed to undergo no deformations, etc. Of course, Lorentz's aether is massless as well. Lorentz tells us, with emphasis, not to

* This, of course, implies the possibility of our following an individual portion or element of charge in its motion,—a subtle point (due to circuital indeterminateness, etc.), which, however, need not detain us here.

† To be contrasted afterwards with his *macroscopic* (or average) equations.

bring in even the notion of a 'force on the aether.' It is true—he adds—that this is against Newton's third law (action = reaction), 'but, as far as I see, nothing compels us to elevate that proposition to a fundamental law of unlimited validity' (*Essay*, p. 28).

But there is no need to keep in mind all these, and similar, remarks and verbal explanations,—especially as the absence of force on the free aether is seen from (II.) at a glance, by putting $\rho = 0$.

> It is perfectly sufficient to state that the basis of Lorentz's theory is entirely contained in the above (*microscopically* valid) equations (I.), (II),* all other things being obtained from these equations by more or less pure deduction, without new hypotheses.†

Notice, in passing, that (I.) is not a complete system in the sense of the word explained in Chap. I. For to trace the electromagnetic history, not only $\mathbf{E}_0$, $\mathbf{M}_0$ for $t = 0$ and for the whole space, but also ρ and $\mathbf{p}$ for all values of t must be given. In (I.) we have, essentially, two vector equations of the first order for three vectors $\mathbf{E}$, $\mathbf{M}$, $\mathbf{p}$, and the formula (II.) does not complete the system, since, on further research, it does not lead to an equation of the form $\partial \mathbf{p}/\partial t = \Omega(\mathbf{E}, \mathbf{M}, \mathbf{p})$,‡ but in the most favourable case to an integral equation extending over a certain *interval* of time, generally finite, but sometimes indefinitely prolonged. But this 'incompleteness' is no disadvantage in (I.), (II.), especially for the purpose of macroscopic treatment, in which consisted Lorentz's main object of constructing these equations.

The equations assembled in (I.), which, together with the formula for the ponderomotive force, have been received into the domain of modern Relativity, as will be seen later, can be easily condensed into a single quaternionic equation. First of all, put

$$\mathbf{B} = \mathbf{M} - \iota \mathbf{E} \tag{11}$$

(where $\iota = \sqrt{-1}$), and call it the **electromagnetic bivector.** Also write, for convenience,

$$l = \iota c t. \tag{12}$$

* These are also the equations of Larmor, who started from the conception of a quasi-rigid aether and deduced the equations in question from the principle of least action. (*Aether and Matter*, Cambridge, 1900.)

† Till he comes to Michelson and Morley's famous interference experiment.

‡ Ω being some space-operator and $\mathbf{E}$, $\mathbf{M}$, $\mathbf{p}$ the instantaneous values of the three vectors or vector-fields.

Then, the first and third, and the second and fourth of (I.) coalesce respectively into the bivectorial equations

$$\frac{\partial \mathbf{B}}{\partial l} + \text{curl}\,\mathbf{B} = \frac{1}{c}\rho\mathbf{p}$$

and

$$\text{div}\,\mathbf{B} = -\iota\rho\,;$$

or, in Hamilton's symbols,

$$\frac{\partial \mathbf{B}}{\partial l} + \mathrm{V}\nabla\mathbf{B} = \frac{1}{c}\rho\mathbf{p},$$

$$\mathrm{S}\nabla\mathbf{B} = -(\nabla\mathbf{B}) = -\text{div}\,\mathbf{B} = \iota\rho.$$

Add up, and remember that the full quaternionic 'product' of the Hamiltonian ∇ and of the bivector **B** is

$$\nabla\mathbf{B} = \mathrm{V}\nabla\mathbf{B} + \mathrm{S}\nabla\mathbf{B}\,;$$

then

$$\frac{\partial \mathbf{B}}{\partial l} + \nabla\mathbf{B} = \rho\left(\iota + \frac{1}{c}\mathbf{p}\right).$$

Next, introduce the operator

$$D = \frac{\partial}{\partial l} + \nabla \tag{13}$$

which will turn out to be of fundamental importance for our subsequent relativistic considerations, and the quaternion

$$C = \rho\left(\iota + \frac{1}{c}\mathbf{p}\right), \tag{14}$$

which we may call the **current-quaternion**. Then the last equation becomes

$$D\mathbf{B} = C. \tag{I. a}$$

Thus, the four vectorial equations in (I.) coalesce into a single quaternionic equation (I. *a*), whose form will be most convenient for relativistic electromagnetism. It is scarcely necessary to say that what we have done here has nothing to do with Relativity. We are not as yet so far. (I. *a*) is simply a formal condensation of the fundamental electronic equations (I.).

What we are mainly concerned with in the present chapter is the macroscopic or average result of these equations and of the force formula (II.). But before passing to consider Lorentz's macroscopic equations, it will be good to dwell here a little upon the exact or

microscopic formulae (I.), (II.), and some of their immediate and most important consequences.

First, as regards the *conservation of energy*, multiply the first of (I.) by $\mathbf{E}$ and the third by $\mathbf{M}$, both times scalarly. Then, remembering that, by (II.), $\rho(\mathbf{Ep}) = (\mathbf{Pp})$, and, by vector algebra,

$$(\mathbf{E}\,\mathrm{curl}\,\mathbf{M}) - (\mathbf{M}\,\mathrm{curl}\,\mathbf{E}) = -\mathrm{div}\,\mathrm{V}\mathbf{EM},$$

the result will be

$$-\frac{\partial u}{\partial t} = (\mathbf{Pp}) + \mathrm{div}\,\mathfrak{P}, \tag{15}$$

where

$$u = \tfrac{1}{2}(E^2 + M^2) \tag{16}$$

and

$$\mathfrak{P} = c\mathrm{V}\mathbf{EM}. \tag{17}$$

Now, $(\mathbf{Pp})$ is the activity of the ponderomotive force or the work done 'by the ether on the electrons' per unit time, and unit volume. Thus, by (15), the principle of conservation of energy will be satisfied for every portion of space, however small, if u is interpreted as the **density**, and at the same time $\mathfrak{P}$ as the **flux**, of **electromagnetic energy**. The possibility of adding to $\mathfrak{P}$ any vector of purely solenoidal distribution need not detain us here. $\mathfrak{P}$ is widely known as the **Poynting vector**, in commemoration of the fact that this vector and the corresponding conception of the flow of energy were first formulated by Poynting (1884). Thus we see that the density and the flux of electromagnetic energy, given by (16) and (17), are in Lorentz's theory precisely as in Maxwell's and Hertz-Heaviside's theory.

Next, as regards the *ponderomotive force* $\mathbf{P}$, in comparison with that of Maxwell as expressed by his electromagnetic stress, use the first and third of the fundamental equations (I.); then (II.) will become

$$\mathbf{P} = \rho\mathbf{E} - \mathrm{V}\mathbf{E}\,\mathrm{curl}\,\mathbf{E} - \mathrm{V}\mathbf{M}\,\mathrm{curl}\,\mathbf{M} - \frac{1}{c}\mathrm{V}\frac{\partial \mathbf{E}}{\partial t}\mathbf{M} - \frac{1}{c}\mathrm{V}\mathbf{E}\frac{\partial \mathbf{M}}{\partial t},$$

or, introducing the Poynting vector,

$$\mathbf{P} = \rho\mathbf{E} - \mathrm{V}\mathbf{E}\,\mathrm{curl}\,\mathbf{E} - \mathrm{V}\mathbf{M}\,\mathrm{curl}\,\mathbf{M} - \frac{1}{c^2}\frac{\partial \mathfrak{P}}{\partial t}. \tag{18}$$

This is the expression of Lorentz's force, equivalent, in virtue of (I.), to the original expression (II.). Now, *Maxwell's ponderomotive force*, per unit volume, is given by

$$\mathbf{P}_{\mathrm{Mx}} = \rho\mathbf{E} - \mathrm{V}\mathbf{E}\,\mathrm{curl}\,\mathbf{E} - \mathrm{V}\mathbf{M}\,\mathrm{curl}\,\mathbf{M}. \tag{19}$$

This is the resultant of Maxwell's well-known electromagnetic stress

$$\mathbf{f}_n = u\mathbf{n} - \mathbf{E}(\mathbf{En}) - \mathbf{M}(\mathbf{Mn}), \tag{20}$$

i.e.

$$\mathbf{P}_{\text{Mx}} = -\mathbf{i}\,\text{div}\,\mathbf{f}_1 - \mathbf{j}\,\text{div}\,\mathbf{f}_2 - \mathbf{k}\,\text{div}\,\mathbf{f}_3, \tag{21}$$

$\mathbf{f}_n$ being the *pressure** per unit area, on a surface element whose unit normal is $\mathbf{n}$, and $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ meaning the same things as $\mathbf{f}_n$ for $\mathbf{n} = \mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively. We do not stop here to show the equivalence of (19) and (21), for we shall have an opportunity to do so later. What concerns us here is the comparison of Lorentz's with Maxwell's ponderomotive force. From (18) and (19) we see that the former is

$$\mathbf{P} = \mathbf{P}_{\text{Mx}} - \frac{1}{c^2}\frac{\partial \mathfrak{P}}{\partial t}. \tag{22}$$

Maxwell's force on the *free aether*, *i.e.* for $\rho = 0$, is, by (19) and the system (I.), which in this case coincides with Maxwell's equations,

$$\mathbf{P}_{\text{Mx}} = \frac{1}{c}\text{V}\mathbf{E}\dot{\mathbf{M}} + \frac{1}{c}\text{V}\dot{\mathbf{E}}\mathbf{M},$$

i.e.

$$\mathbf{P}_{\text{Mx}} = \frac{1}{c^2}\frac{\partial \mathfrak{P}}{\partial t}, \quad \text{for} \quad \rho = 0. \tag{19$_0$}$$

Thus, in a variable field, Maxwell's ponderomotive force on the free aether is, generally, different from zero. The supposed existence of such a force, which has been treated on various occasions by Heaviside, suggested to Helmholtz the argument of his last paper, namely an investigation of the possible motions of the free aether.† On the other hand, Lorentz's force on the free aether is always *nil*, according to his fundamental formula (II.); as has been already remarked, he forbids us even to talk about a force on the aether, since its elements are supposed once and for ever to be immovable. According to (22) the Maxwellian force on the aether is just compensated by Lorentz's supplementary term $-\frac{1}{c^2}\partial\mathfrak{P}/\partial t$. In using the Maxwellian stress $\mathbf{f}_n$ in his theory, Lorentz considers it, of course, as a system of 'merely fictitious tensions' (cf. *Essay*, p. 29). In

* Pressure proper being counted positive, and tension proper negative.

† H. v. Helmholtz, *Folgerungen aus Maxwell's Theorie über die Bewegungen des reinen Aethers*; Berl. Sitzber., July 5, 1893; *Wied. Ann.*, Vol. LIII. p. 135, 1894.

Maxwell's theory the ponderomotive actions observed in electric and magnetic fields were physically accounted for by the tensions and pressures of the aether. But Lorentz, in order to be consistent, avoids considering the 'aether tensions' as something physical, since these would mean forces exerted by the different parts of the aether on one another. Thus, the Maxwellian stress is to him but a convenient instrument for calculation.

Returning to the general case, $\rho \neq 0$, Lorentz's ponderomotive force (II.) may be written, by (22) and (21),

$$\mathbf{P} = -\mathbf{i}\,\mathrm{div}\,\mathbf{f}_1 - \mathbf{j}\,\mathrm{div}\,\mathbf{f}_2 - \mathbf{k}\,\mathrm{div}\,\mathbf{f}_3 - \frac{1}{c^2}\frac{\partial \mathfrak{P}}{\partial t}. \qquad (23)$$

It thus consists of two parts, the first of which is deducible from the Maxwellian stress, while the second, foreign to Maxwell's theory, is given by the negative time-rate of local change of the vector $\mathfrak{P}/c^2$. It is this second term which always compensates the Maxwellian action on the pure aether.

Finally, to obtain Lorentz's *resultant force*

$$\Pi = \int \mathbf{P}\, d\tau$$

on the whole system of electrons (τ being any volume containing all the electrons), use the expression (23), and observe that

$$\int \mathrm{div}\,\mathbf{f}_i\, d\tau = \int (\mathbf{n}\mathbf{f}_i)\, d\sigma, \quad i = 1, 2, 3,$$

where $\mathbf{n}$ is the outward unit normal of the surface σ enclosing the region τ. Also remember that

$$\mathbf{i}(\mathbf{f}_1\mathbf{n}) + \mathbf{j}(\mathbf{f}_2\mathbf{n}) + \mathbf{k}(\mathbf{f}_3\mathbf{n}) = \mathbf{f}_n, \qquad (24)$$

since the Maxwellian stress is irrotational or self-conjugate. Then the result will be

$$\Pi = \int \mathbf{f}_n\, d\sigma - \frac{d}{dt}\int \frac{1}{c^2}\mathfrak{P}\, d\tau, \qquad (25)$$

σ being supposed fixed in the aether, *i.e.* relatively to the framework S in which the fundamental equations are to be valid. Formula (25) states simply the same thing for the whole system, contained in τ, which is expressed by (23) for each of its elements. Of course, in passing from (23) to (25), the continuity of the vector $\mathbf{f}_n$ (or at least of its components normal to surfaces of discontinuity, if there be any)

has been tacitly assumed throughout τ.* The last formula, again, may be written:

$$\Pi = \Pi_{Mx} - \frac{d}{dt}\int \frac{1}{c^2} \mathfrak{P}\, d\tau,$$

which needs no further explanation. Now, as the mathematicians say, let σ expand to infinity, or at least so that, E, M decreasing in the usual way as $1/r^2$, the surface integral may vanish. Then

$$\Pi_{Mx} = 0,$$

while

$$\Pi = -\frac{d\mathbf{G}}{dt} \tag{26}$$

where the vector $\mathbf{G}$ is defined by

$$\mathbf{G} = \frac{1}{c^2}\int \mathfrak{P}\, d\tau, \tag{27}$$

and is called the **electromagnetic momentum.**

Thus Maxwell's resultant force is strictly *nil*, satisfying Newton's third law (*actio est par reactioni*), while Lorentz's resultant force is generally different from zero, against the third law,—a result which has been already stated in a slightly different form. Thus Maxwell's theory, admitting an action on the pure aether, did, while Lorentz's theory, denying it, does not satisfy Newton's third law. But, as was observed by Lorentz himself, there is nothing to compel us to universalize that law of Newtonian mechanics. At first, Poincaré tried to use this as an argument against Lorentz's theory;† but he soon gave it up. This was to be only one of a whole series of sacrifices, and not the greatest one, made by modern physicists.

Similarly, the *resultant moment* of the ponderomotive forces,

$$\Omega = \int \mathrm{V}\mathbf{r}\mathbf{P}\, d\tau, \tag{28}$$

where $\mathbf{r}$ is the vector drawn to any point of the field from a point O fixed in the aether, or fixed relatively to S, may be easily put into the form

$$\Omega = \int \mathrm{V}\mathbf{r}\mathbf{f}_n d\sigma - \frac{d}{dt}\int \frac{\mathrm{V}\mathbf{r}\mathfrak{P}}{c^2}\, d\tau.$$

*The treatment of possible exceptions to this assumption, as electromagnetic surfaces of discontinuity or *waves* properly so called [which exceptions seem to be overlooked by the leading electronists, who claim for (25) general validity], need not detain us here.

† H. Poincaré, *Arch. Néerland.*, Vol. V.; 1900.

Thus, for the whole space, and with the usual assumption as to the behaviour of E, M at infinity,

$$\Omega_{Mx}=0,$$

and

$$\Omega=-\frac{d\mathbf{H}}{dt}, \tag{29}$$

where

$$\mathbf{H}=\frac{1}{c^2}\int \mathrm{V}\mathbf{r}\mathfrak{P}\,d\tau$$

is called the *electromagnetic moment of momentum.* Its analogy to the ordinary, mechanical, moment of momentum

$$\Sigma m\mathrm{V}\mathbf{rv}$$

is obvious. So is also the analogy of the above $\mathbf{G}$ with the ordinary momentum

$$\Sigma m\mathbf{v}$$

and the corresponding interpretation of (26) and (29). Both $\mathbf{G}$ and $\mathbf{H}$ are so constructed as if the aether contained (electromagnetic) momentum in each of its elements amounting to

$$\mathbf{g}=\frac{1}{c^2}\mathfrak{P}=\frac{1}{c}\mathrm{V}\mathbf{EM} \tag{30}$$

per unit volume.

So much as regards the chief consequences of the fundamental formulae (I.) and (II.).

Now for Lorentz's *macroscopic* equations. These are obtained from (I.), (II.) by averaging over 'physically infinitesimal' regions of space. Lorentz calls a length l 'physically infinitesimal' (in distinction from what is called 'mathematically infinitesimal') if the values of any observable magnitude obtaining in two points distant l from one another are sensibly equal to, *i.e.* indiscernible from, one another. Molecular, and, *a fortiori*, electronic, dimensions and mutual distances of molecules constituting a ponderable medium, are assumed to be small fractions of l. Let ψ be any magnitude, scalar or vectorial. Round a point P draw a sphere of physically infinitesimal radius; let τ be the volume of this sphere. Then

$$\frac{1}{\tau}\int \psi d\tau$$

is called the 'mean value of ψ at P,' and is denoted by $\overline{\psi}$. If ψ be any of the magnitudes involved in the fundamental (microscopic)

equations, as for instance ρ or $\mathbf{M}$, then $\overline{\psi}$ is what is macroscopically observable.

We cannot reproduce here the details of the process of averaging based upon the above fundamental notion,* but shall simply write down the resulting macroscopic equations, limiting ourselves to the case of a *perfectly transparent* (*i.e.* non-conducting), *non-magnetic* ponderable medium, and leaving out of account dispersion. We must, however, explain first the meaning of the symbols involved in these equations.

Assuming that the molecules of the ponderable medium or body contain electrons,† to which belong certain positions of equilibrium within the individual molecules, Lorentz supposes their displacements from these positions, $\mathfrak{q}$, and their velocities relative to the corresponding molecule,

$$\dot{\mathfrak{q}} = d\mathfrak{q}/dt,$$

to be infinitesimal. In other words, he neglects the squares and products of $\mathfrak{q}$, $\dot{\mathfrak{q}}$, or any of their components in presence of their first powers. Notice that the only part played by the molecules of ponderable matter consists here in restraining the electrons, *i.e.* in keeping them near certain positions. For, as has already been remarked, one of Lorentz's fundamental assumptions is, that matter by itself, apart from electricity, behaves like the free aether, its presence having no influence whatever upon the electromagnetic field.

Let e be the charge of an electron which has experienced the displacement $\mathfrak{q}$, as explained above. Then Lorentz brings in the notion of *electrical moment*, not unfamiliar to older theories, defining this vector to be, per unit volume, the average of $e\mathfrak{q}$, *i.e.*

$$\overline{e\mathfrak{q}}.$$

Taking the sum of this and of the average of our above $\mathbf{E}$, Lorentz introduces the macroscopic vector

$$\mathfrak{E} = \overline{\mathbf{E}} + \overline{e\mathfrak{q}} \tag{31}$$

*See Sections II. and IV. of Lorentz's *Essay*, or his article in *Encykl. d. math. Wiss.*, Vol. V_2, pp. 200 *et seq.*; Leipzig, 1904.

†Viz. 'polarization-electrons,' and leaving out of account circling or 'magnetization-' and free or 'conduction-electrons.'

which he calls the *dielectric polarization.** Thus, in the free aether $\mathfrak{E}$ reduces to $\overline{\mathbf{E}}$, and generally $\mathfrak{E}$ is what Maxwell called the dielectric displacement.

Next, the macroscopic *magnetic force* is defined to be the average of our above **M**, *i.e.* $\overline{\mathbf{M}}$, instead of which, however, we shall write shortly **M**.

Finally, the macroscopic *electric force* is introduced, being defined as the average of $\overline{\mathbf{E}'}$, *i.e.* of the ponderomotive force per unit charge, as given by the formula (10). Instead of $\overline{\mathbf{E}'}$ we shall, again, write more conveniently $\mathbf{E}'$. Thus Lorentz's macroscopic electric force will be

$$\mathbf{E}' = \overline{\mathbf{E}} + \frac{1}{c}\overline{\mathrm{V}\mathbf{pM}}. \tag{32}$$

Notice that here **p** means the resultant velocity of an electron, *i.e.* the vector sum of its velocity relatively to the molecule in question and of the velocity of the ponderable body as a whole, say **v**, 'relatively to the aether,' so that $\mathbf{p} = \dot{\mathbf{q}} + \mathbf{v}$.

With these meanings of the symbols, Lorentz's macroscopic equations for a *transparent*, *non-magnetic*, ponderable body, *moving with constant*† *velocity* **v** 'through the stagnant aether,' *i.e.* relatively to the framework *S*, are as follows (*Essay*, p. 76):

$$\left.\begin{aligned}
&\frac{\partial \mathfrak{E}}{\partial t} = c\,.\operatorname{curl}\mathbf{M}'\,; \quad \operatorname{div}\mathfrak{E} = 0\\
&\frac{\partial \mathbf{M}}{\partial t} = -c\,.\operatorname{curl}\mathbf{E}'\,; \quad \operatorname{div}\mathbf{M} = 0\\
&\mathbf{M}' = \mathbf{M} - \frac{1}{c}\mathrm{V}\mathbf{vE}'\\
&K\mathbf{E}' = \mathfrak{E} + \frac{1}{c}\mathrm{V}\mathbf{vM}.
\end{aligned}\right\} \tag{33}$$

Here the system of coordinates involved in div and curl, is *rigidly attached to the ponderable body*, thus sharing in its motion through the aether. But the time *t* is the same as in the fundamental equations (I.); obviously, therefore, $\partial/\partial t$ is the time rate of change (for constant values of those coordinates, *i.e.*) *at a fixed point of the body*, not of the aether or of *S*.

*The above $\mathfrak{E}$ is Lorentz's $\mathfrak{D}$.

† *Constant* in space and time, that is to say for a body having a uniform purely translational, rectilinear motion.

The second of (33) is an obvious expression of the (assumed) absence of macroscopic charge, *i.e.* of $\bar{\rho}=0$. In the more general case of a sensibly charged body we should have div $\mathfrak{E}=\bar{\rho}$, where $\bar{\rho}$ is the observable density. As to K, appearing in the last of (33), it is a linear vector operator in crystalline, and a simple scalar coefficient in isotropic bodies, known as the 'dielectric constant' or *permittivity*, and depending in a complicated way on the distributional properties of the electrons. The numerical value of K in an isotropic, and its principal values, K_1, K_2, K_3 in a crystalline body, are not constant, of course, but vary with the period T of the incident light- or, generally, electromagnetic oscillations. However, to avoid unnecessary complication, we may think here of the simple case of homogeneous light, of a particular kind (colour). Then K, or K_1, K_2, K_3, are constants, whose numerical values are to be considered as deduced from the observable refractive properties of the body with regard to light of that particular kind. In case of isotropy we have to write $K=n^2$, if n be the corresponding index of refraction.*

Notice that (33) contains, besides the solenoidal conditions for $\mathfrak{E}$ and $\mathbf{M}$, four vector equations for as many vectors,

$$\mathfrak{E}, \mathbf{M}, \mathbf{E}', \mathbf{M}',$$

the velocity of motion $\mathbf{v}$ being given. And since the differential equations are of the first order with regard to t, the electromagnetic history of the whole medium is determined by its initial state, say, by $\mathfrak{E}_0$, $\mathbf{M}_0$ given for $t=0$.

It must be kept in mind that, to obtain the system of equations (33) from the fundamental ones, Lorentz has consciously neglected not only various small terms concerning the minute influence of electrons, but also *all terms of second order in* β, or, to put it shortly, *all* β^2*-terms*, where

$$\beta=\frac{v}{c}\,.$$

This is especially true of the fifth of (33), which has been obtained from the more exact formula $\mathbf{M}'=\mathbf{M}-\mathrm{V}\mathbf{v}\overline{\mathbf{E}}/c$ by writing $\mathbf{E}'$ instead of $\overline{\mathbf{E}}$, and thus [cf. (32)] omitting $\mathrm{V}\mathbf{v}\overline{\mathrm{V}\mathbf{p}\mathbf{M}}/c^2$, which is a β^2-term.

* As to dispersion, which need not detain us here, it can be accounted for in the well-known way by attributing to the body (or to its molecules) one or more internal, 'natural periods,' and, to introduce these, plenty of opportunities are offered by the hypothesis of the electronic structure of molecules and atoms.

Let us now consider some of the most important consequences flowing from the above system of macroscopic equations.

First of all, as the reader may easily find by himself,* they give the right value for the dragging coefficient, viz. sensibly Fresnel's coefficient, $\kappa = 1 - \frac{1}{n^2}$. This, in fact, is a consequence of (33), when β^2-terms are neglected and when dispersion is not taken into account. For a dispersive medium that value of the index of refraction is to be taken which corresponds to the 'relative' period of oscillation, T',—a concept to be explained further on. This gives a slight correction term,—$n^{-1}T\partial n/\partial T$ (*Essay*, p. 101), where n is the refractive index of the medium corresponding to the 'absolute' period T, *i.e.* the period of the oscillations emitted by the source, say, in Fizeau's experiment. Thus, Lorentz's formula is

$$\kappa = 1 - \frac{1}{n^2} - \frac{1}{n} T \frac{\partial n}{\partial T}. \tag{Lor}$$

For water, at 18° C., and for sodium light, this becomes

$$\kappa = 0{\cdot}451, \tag{Lor}$$

whereas Fresnel's value, and that obtained experimentally by Michelson and Morley, have been 0·438 and 0·434 ± 0·02 respectively. Thus Lorentz's dragging coefficient agrees with the experimental value (MM) quite as well as Fresnel's, especially if the 'possible error of ± 0·02' be taken into account. In a word, *Lorentz's equations give the right value of the dragging coefficient.* And, from what has been said previously, it can be argued that these equations will also give correct results for all *first order* phenomena.

Next, putting $\mathbf{v} = 0$, we see at once that (33) become

$$\left.\begin{aligned} &\frac{\partial \mathfrak{E}}{\partial t} = c\,.\,\text{curl}\,\mathbf{M}; \quad \text{div}\,\mathfrak{E} = 0 \\ &\frac{\partial \mathbf{M}}{\partial t} = -c\,.\,\text{curl}\,\mathbf{E}; \quad \text{div}\,\mathbf{M} = 0 \\ &\mathfrak{E} = K\mathbf{E} \end{aligned}\right\} \tag{33_0}$$

that is to say, *identical with Maxwell's equations, for a stationary* (non-magnetic) *medium*, (1), p. 24. Taking account of

* Proceeding, *mutatis mutandis*, similarly as in **Note 3**, concerning (HH). Another, more simple, method of obtaining the dragging coefficient is to apply Lorentz's 'theorem of corresponding states,' to be considered later.

magnetization-electrons, we would have, in the second and third equation, $\mathfrak{M}$ instead of **M**, where $\mathfrak{M}=\mu\mathbf{M}$, μ being the permeability.

This is a very satisfactory result, for, as already mentioned, Maxwell's equations for stationary media, agreeing fully with experiment, have been able to stand even the severe criticism of the modern relativists, who have adopted them without the slightest modification whatever. 'Stationary' means, of course, in Lorentz's theory, fixed relatively to the aether.

In order to exhibit the properties of his equations, (33), in the general case of any constant **v**, *i.e.* for a material medium having any uniform motion of rectilinear translation relative to the aether, Lorentz transforms these equations by introducing instead of the time t a new variable of very remarkable properties. This, the so-called 'local time,' which was to become one of the most immediate forerunners of Einstein's relativistic theory, deserves a rather more extended treatment. It will occupy our attention in the next chapter.

NOTES TO CHAPTER II.

Note 1 (to page 28). Let σ be a surface of electromagnetic discontinuity of first order, for example; that is to say, the vectors **E**, **M** being themselves continuous across σ, let their space- and time-derivatives of first order be different in absolute value and direction on the two sides of the surface. Call one of its sides 1, and the other 2; draw the normal unit vector **n** from 1 towards 2, and denote by $[\alpha]$ the jump of any magnitude α, *i.e.* the difference $\alpha_2-\alpha_1$. Then the so-called **identical conditions**, to be fulfilled in any case, are

$$[\operatorname{div}\mathbf{E}]=(\mathbf{ne})\,;\quad[\operatorname{curl}\mathbf{E}]=\mathrm{V}\mathbf{ne}\,;\tag{a}$$

and the **kinematical condition of compatibility**, valid under the supposition that the surface is neither being split into two or more nor dissolved, is

$$\left[\frac{\partial\mathbf{E}}{\partial t}\right]=-\mathfrak{b}\mathbf{e},\tag{b}$$

e being the same vector as in (a), characterizing the electrical discontinuity, and $\mathfrak{b}$ (an independent scalar) the **velocity of propagation** of σ, counted positively along **n**. Both **e** and $\mathfrak{b}$ remain so far indeterminate, in numerical value and direction. Similarly, for the magnetic discontinuity,

$$[\operatorname{div}\mathbf{M}]=(\mathbf{nm}),\quad[\operatorname{curl}\mathbf{M}]=\mathrm{V}\mathbf{nm},\tag{a_1}$$

$$\left[\frac{\partial\mathbf{M}}{\partial t}\right]=-\mathfrak{b}\mathbf{m},\tag{b_1}$$

$\mathbf{m}$ being a new vector and $\mathfrak{b}$ the same scalar as above, since the electric and magnetic discontinuities are supposed not to part from one another. (For the deduction of the above conditions see J. Hadamard's *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Paris, 1903, or, in vectorized form, my book on *Vectorial Mechanics*, London, Macmillan & Co., 1913; also *Annalen der Physik*, Vol. XXVI., 1908, p. 751 and Vol. XXIX., 1909, p. 523.)

If $\mathbf{e}$, $\mathbf{m}$ are normal to σ, we have a **longitudinal**, and if tangential, a **transversal** discontinuity.

So far everything has been independent of any electromagnetic connections. Now use Maxwell's equations (4), with (4_1); since they are valid on both sides of σ, we have also

$$\left[\frac{\partial \mathbf{E}}{\partial t}\right]=c[\mathrm{curl}\,\mathbf{M}], \text{ etc.,}$$

and, using (*a*), (*b*) with their magnetic analogues,

$$\left.\begin{array}{ll} \dfrac{\mathfrak{b}}{c}\mathbf{e}=\mathrm{V}\mathbf{mn}\,; & \dfrac{\mathfrak{b}}{c}\mathbf{m}=\mathrm{V}\mathbf{ne} \\ (\mathbf{mn})=0\,; & (\mathbf{en})=0. \end{array}\right\} \qquad (c)$$

Notice that if $\mathfrak{b}$ does not vanish, *i.e.* if there is propagation at all, the second pair of equations becomes superfluous, since it then follows identically from the first pair. Now, eliminating $\mathbf{m}$ from the first pair of (*c*), we have

$$\frac{\mathfrak{b}^2}{c^2}\mathbf{e}=\mathrm{V}\mathbf{n}\mathrm{V}\mathbf{en}=\mathbf{e}-\mathbf{n}(\mathbf{en}),$$

$\mathbf{n}$ being a *unit* vector. But $(\mathbf{en})=0$; hence

$$\frac{\mathfrak{b}^2}{c^2}\mathbf{e}=\mathbf{e},$$

and similarly

$$\frac{\mathfrak{b}^2}{c^2}\mathbf{m}=\mathbf{m}.$$

Thus, if *e*, *m* do not vanish, *i.e.* if there is at all a discontinuity,

$$\mathfrak{b}=\pm c\,; \qquad (d)$$

that is to say, each element $d\sigma$ of the wave is propagated normally to itself with the velocity *c*. Q.E.D.

Notice that the sign of $\mathfrak{b}$, left undetermined in (*d*), due to the quadratic result of elimination, may be defined uniquely by means of the original pair of equations (*c*), which are linear in $\mathfrak{b}$. In fact, multiply the first scalarly by $\mathbf{e}$ (or the second by $\mathbf{m}$), then

$$\mathfrak{b}=s(\mathbf{e}\mathrm{V}\mathbf{mn})=s(\mathbf{n}\mathrm{V}\mathbf{em}),$$

where s is a positive scalar, namely c/e^2. Thus, if $\mathbf{n}$, $\mathbf{e}$, $\mathbf{m}$ is a right-handed system, like the usual $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ then $\mathfrak{b}$ is positive, *i.e.* the sense of

propagation is that of $\mathbf{n}$, and if $\mathbf{n}, \mathbf{e}, \mathbf{m}$ is left-handed, then the propagation is along $-\mathbf{n}$. Thus, the sense of propagation coincides always with that of the vector

$$\mathrm{V}\mathbf{em}.$$

If $\mathbf{e}$ points upwards and $\mathbf{m}$ to the right, then the wave is propagated forwards. Notice the similarity with the sense of the flux of energy, or the Poynting vector, in relation to $\mathbf{E}$, $\mathbf{M}$,

$$\mathfrak{P} = c\,\mathrm{V}\mathbf{EM}.$$

Finally, notice, in passing, that by the first pair of (c),

$$e^2 = m^2,$$

similarly to the known characteristic, $E^2 = M^2$, of the usual 'pure' waves.

The above results may easily be extended to waves of discontinuity of any order.

Note 2 (to page 30). Take as a surface element the parallelogram constructed on two coinitial line elements $\mathbf{a}$, $\mathbf{b}$, composed always of the same particles, so that, $\mathbf{n}$ being its positive normal,

$$\mathbf{n}\,d\sigma = \mathrm{V}\mathbf{ab}.$$

Write, generally, $\mathbf{R}$ for $\mathfrak{E}$ or $\mathfrak{M}$. Then the induction through $d\sigma$ will be given by the volume of the parallelopiped $\mathbf{R}$, $\mathbf{a}$, $\mathbf{b}$, *i.e.*

$$(\mathbf{Rn})\,d\sigma = (\mathbf{R}\mathrm{V}\mathbf{ab}).$$

The current through $d\sigma$, say $(\mathbf{pn})d\sigma$, being the rate of change of this induction, is

$$(\mathbf{pn})\,d\sigma = (\dot{\mathbf{R}}\mathbf{n})\,d\sigma + (\mathbf{R}\mathrm{V}\dot{\mathbf{a}}\mathbf{b}) + (\mathbf{R}\mathrm{V}\mathbf{a}\dot{\mathbf{b}}), \tag{a}$$

where the dots stand for *individual* variation. Thus

$$\dot{\mathbf{R}} = \frac{d\mathbf{R}}{dt} = \frac{\partial \mathbf{R}}{\partial t} + (\mathbf{v}\nabla)\mathbf{R}, \tag{b}$$

and [*Vectorial Mechanics*, Chap. V., formula (75)]

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (\mathbf{a}\nabla)\mathbf{v}\,; \quad \dot{\mathbf{b}} = (\mathbf{b}\nabla)\mathbf{v}.$$

Now, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ being the usual right-handed system of mutually normal unit vectors, take a rectangular $d\sigma$, say

$$\mathbf{a} = \mathbf{j}\,dy, \quad \mathbf{b} = \mathbf{k}\,dz,$$

and, consequently,

$$\mathbf{n} = \mathbf{i}, \qquad d\sigma = dy\,.\,dz.$$

Then

$$\dot{\mathbf{a}} = dy\frac{\partial \mathbf{v}}{\partial y}\,; \quad \dot{\mathbf{b}} = dz\frac{\partial \mathbf{v}}{\partial z},$$

so that the sum of the last two terms in (a) will be

$$\left(\mathbf{R}\mathrm{V}\frac{\partial \mathbf{v}}{\partial y}\mathbf{k}\right)d\sigma+\left(\mathbf{R}\mathrm{V}\mathbf{j}\frac{\partial \mathbf{v}}{\partial z}\right)d\sigma,$$

or, per unit area,

$$R_1\left(\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-R_2\frac{\partial v_1}{\partial y}-R_3\frac{\partial v_1}{\partial z}=R_1\operatorname{div}\mathbf{v}-(\mathbf{R}\nabla)v_1\,;$$

hence, substituting (b) in the first term of (a) and remembering that

$$(\mathbf{pn})=(\mathbf{pi})=p_1,\quad (\mathbf{Rn})=R_1,$$

$$p_1=\frac{\partial R_1}{\partial t}+(\mathbf{v}\nabla)R_1+R_1\operatorname{div}\mathbf{v}-(\mathbf{R}\nabla)v_1,$$

with similar expressions for p_2, p_3 if $d\sigma$ be taken normal to $\mathbf{j}$ or $\mathbf{k}$ respectively. Thus the resultant current will be

$$\mathbf{p}=\text{current}\,(\mathbf{R})=\frac{\partial \mathbf{R}}{\partial t}+(\mathbf{v}\nabla)\mathbf{R}-(\mathbf{R}\nabla)\mathbf{v}+\mathbf{R}\operatorname{div}\mathbf{v},$$

or

$$\mathbf{p}=\text{current}(\mathbf{R})=\frac{\partial \mathbf{R}}{\partial t}+\mathbf{v}\operatorname{div}\mathbf{R}+\operatorname{curl}\mathrm{V}\mathbf{R}\mathbf{v}, \qquad (c)$$

which is the required formula.

In the simplest case, considered on p. 33, in which the material medium moves as a whole with purely translational velocity $\mathbf{v}=v\mathbf{i}$, we have to take only the first term of (a), so that in this case

$$\mathbf{p}=\frac{d\mathbf{R}}{dt}=\frac{\partial \mathbf{R}}{\partial t}+(\mathbf{v}\nabla)\mathbf{R}=\frac{\partial \mathbf{R}}{\partial t}+v\frac{\partial \mathbf{R}}{\partial x}. \qquad (c_1)$$

Note 3 (to page 34). Take $\mathbf{E}$, etc., proportional to an exponential function of the argument

$$g(x-\mathfrak{v}t),$$

where g is an imaginary constant, as usual. Then

$$\frac{\partial}{\partial t}=-g\mathfrak{v}\,;\quad \nabla=\mathbf{i}\frac{\partial}{\partial x}=g\mathbf{i},$$

and, consequently, curl $=\mathrm{V}\nabla=g\mathrm{V}\mathbf{i}$. Introducing this in the equations (Mx), remembering that $\mathbf{v}=v\mathbf{i}$ and omitting the common factor g, we obtain at once

$$-\frac{\mathfrak{v}}{c}\mathfrak{E}=\mathrm{V}\mathbf{i}\mathbf{M},$$

$$\frac{\mathfrak{v}}{c}\mathfrak{M}=\mathrm{V}\mathbf{i}\left\{\mathbf{E}-\frac{v}{c}\mathrm{V}\mathbf{i}\mathfrak{M}\right\}$$

$$=\mathrm{V}\mathbf{i}\mathbf{E}-\frac{v}{c}\left\{\mathbf{i}(\mathfrak{M}\mathbf{i})-\mathfrak{M}\right\};$$

but div $\mathfrak{M}=0$ gives in the present case $(\mathfrak{M}\mathbf{i})=0$. Thus

$$\mathfrak{v}\mathfrak{E}=c\mathrm{V}\mathbf{M}\mathbf{i},$$

$$(\mathfrak{v}-v)\mathfrak{M}=c\mathrm{V}\mathbf{i}\mathbf{E},$$

and, the medium being isotropic,

$$(\mathfrak{v}-v)\mathbf{M}=\frac{c}{\mu}\mathrm{V}\mathbf{i}\mathbf{E},$$

$$\mathfrak{v}\mathbf{E}=\frac{c}{K}\mathrm{V}\mathbf{M}\mathbf{i}.$$

Eliminate $\mathbf{E}$, remembering that $(\mathbf{M}\mathbf{i})=0$; then the result will be

$$\mathfrak{v}(\mathfrak{v}-v)\mathbf{M}=\frac{c^2}{K\mu}\mathbf{M}=\mathfrak{v}'^2\mathbf{M},$$

where $\mathfrak{v}'$ would be the velocity of propagation, if the medium were stationary in S. Thus

$$\mathfrak{v}(\mathfrak{v}-v)=\mathfrak{v}'^2,$$

and, the sense of propagation being that of $\mathbf{v}$,

$$\mathfrak{v}=\tfrac{1}{2}v+\sqrt{\mathfrak{v}'^2+v^2/4},$$

which is the required formula.

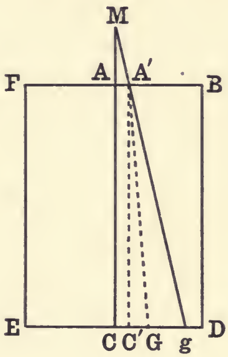
Note 4 (to page 39). To spare me trouble and to give the reader a sample of Fresnel's charming manner of exposition, I quote here simply the closing passages of his letter to Arago (*loc. cit.* pp. 633-636), in which he treats in a masterly manner the *water-telescope experiment*, both on the corpuscular and on the undulatory theory of light:

'Je terminerai cette lettre par une application de la même théorie à l'expérience proposée par Boscovich, consistant à observer le phénomène de l'aberration avec des lunettes remplies d'eau, ou d'un autre fluide beaucoup plus réfringent que l'air, pour s'assurer si la direction dans laquelle on aperçoit une étoile peut varier en raison du changement que le liquide apporte dans la marche de la lumière. Je remarquerai d'abord qu'il est inutile de compliquer de l'aberration le résultat que l'on cherche, et qu'on peut aussi bien le déterminer en visant un objet terrestre qu'une étoile. Voici, ce me semble, la manière la plus simple et la plus commode de faire l'expérience.'

'Ayant fixé à la lunette même, ou plutôt au microscope *FBDE* [figure 2 of Fresnel's letter], le point de mire *M*, situé dans le prolongement de son axe optique *CA*, on dirigerait ce système perpendiculairement à l'écliptique, et, après avoir fait l'observation dans un sens, on le retour-

nerait bout pour bout, et l'on ferait l'observation en sens contraire. Si le mouvement terrestre déplaçait l'image du point M par rapport au fil de l'oculaire, on la verrait de cette manière tantôt à droite et tantôt à gauche du fil.'

'Dans le système d'émission, il est clair, comme Wilson l'a déjà remarqué, que le mouvement terrestre ne doit rien changer aux apparences du phénomène. En effet, il résulte de ce mouvement que le rayon partant de M doit prendre, pour passer par le centre de l'objectif, une direction MA' telle que l'espace AA' soit parcouru par le globe dans le même intervalle de temps que la lumière emploie à parcourir MA', ou MA (à cause de la petitesse de la vitesse de la terre relativement à celle



de la lumière). Représentant par v la vitesse de la lumière dans l'air, et par t celle de la terre [*i.e.* our c and v respectively], on a donc :

$$MA : AA' :: v : t \quad \text{ou} \quad \frac{AA'}{MA} = \frac{t}{v};$$

c'est le sinus d'incidence. v' étant la vitesse de la lumière dans le milieu plus dense que contient la lunette [v' is our c/n], le sinus de l'angle de réfraction $C'A'G$ sera égal à $\frac{t}{v'}$; on aura donc $C'G = A'C'\frac{t}{v'}$; d'où l'on tire la proportion

$$C'G : A'C' :: t : v'.$$

Par conséquent le fil C' de l'oculaire placé dans l'axe optique de la lunette arrivera en G en même temps que le rayon lumineux qui a passé par le centre de l'objectif.'

So far the corpuscular or emission theory. Again :

'La théorie des ondulations conduit au même résultat. Je suppose, pour plus de simplicité, que le microscope est dans le vide. d et d' étant les vitesses de la lumière dans le vide et dans le milieu que contient la

lunette, on trouve pour le sinus de l'angle d'incidence AMA', $\frac{t}{d}$, et pour celui de l'angle de réfraction $C'AG$, $\frac{td'}{d^2}$. Ainsi, indépendamment du déplacement des ondes dans le sens du mouvement terrestre,

$$C'G = A'C'\frac{td'}{d^2}.$$

Mais la vitesse avec laquelle ces ondes sont entraînées par la partie mobile du milieu dans lequel elles se propagent est égale à

$$t\left(\frac{d^2-d'^2}{d^2}\right)$$

[*i.e.* in our notation $v\left(1-\frac{1}{n^2}\right)$]; donc leur déplacement total Gg, pendant le temps qu'elles emploient à traverser la lunette, est égal à

$$\frac{A'C'}{d'}t\left(\frac{d^2-d'^2}{d^2}\right);$$

ainsi

$$C'g = A'C'.t\left(\frac{d'}{d^2}+\frac{d^2-d'^2}{d'd^2}\right) = A'C'.t\left(\frac{d^2}{d'd^2}\right) = A'C'\cdot\frac{t}{d'}.$$

On a donc la proportion $C'g : A'C' :: t : d'$; par conséquent l'image du point M arrivera en g en même temps que le fil du micromètre. Ainsi les apparences du phénomène doivent toujours rester les mêmes quel que soit le sens dans lequel on tourne cet instrument. Quoique cette expérience n'ait point encore été faite, je ne doute pas qu'elle ne confirmât cette conséquence, que l'on déduit également du système de l'émission et de celui des ondulations.'

Note 5 (to page 42). Stokes' theory of aberration ('On the Aberration of Light,' *Phil. Mag.*, Vol. XXVII., 1845, p. 9, reprinted in *Math. and Phys. Papers*, Vol. I. p. 134) was based on the assumption that the aether surrounding the earth is dragged by this planet in its annual motion, in such a way that the velocity of the aether relative to the earth is *nil near its surface*, and, increasing gradually, becomes equal and opposite to the earth's orbital velocity at very considerable distances from our planet. It is obvious that this hypothesis led at once to a rigorous independence of purely terrestrial optical phenomena from the earth's annual motion. But in order to explain correctly astronomical aberration, Stokes had to assume that the aether's motion, between the earth and the 'fixed' stars, is purely *irrotational*, which assumption could not be reconciled with the absence of sliding over the earth's surface, so long as the aether was regarded as incompressible. It is true that this difficulty, as has been shown by Planck, can be overcome by giving up the incompressibility, namely by supposing the aether to be condensed around the earth and the celestial bodies, as if it were subjected to

gravitation and behaved more or less like a gas. But the condensation around the earth, required to reduce the sliding to, say, one half per cent. of the earth's orbital velocity, would be something like e^{11}, *i.e.* corresponding to a density of the aether near the earth about 60,000 times as great as its density in celestial space. Now, it is certainly difficult to admit that the velocity of light is not to any sensible extent altered by this enormous condensation of the aether around the earth.

Particulars concerning the discussion of this most interesting subject will be found in Lorentz's book on *Theory of Electrons* (Chap. V.), and in his original paper on 'Stokes' Theory of Aberration in the Supposition of a Variable Density of the Aether,' *Amsterdam Proceedings*, 1898-1899, p. 443, reprinted in *Abhandlungen üb. theor. Physik*, Vol. I. p. 454.

CHAPTER III.

THEOREM OF CORRESPONDING STATES. SECOND ORDER DIFFICULTIES. THE CONTRACTION HYPOTHESIS. LORENTZ'S GENERALIZED THEORY.

LET us return to Lorentz's macroscopic equations, for a material medium moving relatively to the aether with uniform velocity **v**,

$$\left.\begin{aligned} &\frac{\partial\mathfrak{E}}{\partial t}=c\,.\,\mathrm{curl}\,\mathbf{M}'\,; \qquad \mathrm{div}\,\mathfrak{E}=0 \\ &\frac{\partial\mathbf{M}}{\partial t}=-c\,.\,\mathrm{curl}\,\mathbf{E}'\,; \quad \mathrm{div}\,\mathbf{M}=0 \\ &\mathbf{M}'=\mathbf{M}-\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}' \\ &K\mathbf{E}'=\mathfrak{E}+\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{M}. \end{aligned}\right\} \qquad (\mathrm{L})$$

In the simplest case of a medium fixed in the aether, *i.e.* for $v=0$, these, as already noticed, become identical with Maxwell's equations for a stationary dielectric,

$$\left.\begin{aligned} &\frac{\partial\mathfrak{E}}{\partial t}=c\,.\,\mathrm{curl}\,\mathbf{M}\,; \qquad \mathrm{div}\,\mathfrak{E}=0 \\ &\frac{\partial\mathbf{M}}{\partial t}=-c\,.\,\mathrm{curl}\,\mathbf{E}\,; \quad \mathrm{div}\,\mathbf{M}=0 \\ &\qquad\qquad \mathfrak{E}=K\mathbf{E}. \end{aligned}\right\} \qquad (\mathrm{L}_0)$$

In order to exhibit the properties of the more general equations (L), Lorentz introduces instead of the 'universal time,' as he calls t, a new variable t', which will now be explained.

Let O' be a point fixed in the material body, chosen arbitrarily but once and for ever as the origin of coordinates, x', y', z', measured

along axes rigidly attached to the body. From O' draw to any individual point of the body $P'(x', y', z')$ the vector $\mathbf{r}'$, so that the three Cartesian coordinates are condensed in

$$\mathbf{r}' = \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z'.$$

Let us call the framework of reference rigidly attached to the body the *system* S'. For comparison and to impress better upon your mind the meaning of $\mathbf{r}'$, take also an initial point O fixed in the aether, *i.e.* relatively to the system S, and draw from O to P' the vector $\mathbf{r}$, or in semi-Cartesian expansion, using the same unit vectors as above,*

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z.$$

If O' is taken to coincide with O at the instant $t = 0$, we have simply

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t.$$

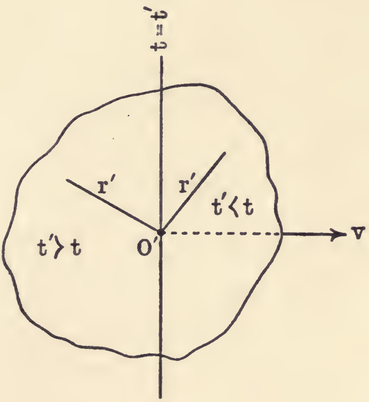


Fig. 6.

Remember that the equations (L) hold for t and $x', y'\ z'$ (not x, y, z) as independent variables, or, more shortly, for

$$\mathbf{r}', \ t.$$

This fixes the meaning of curl, div and $\partial/\partial t$, as already mentioned in Chap. II. As regards the curls and divergences, they are, of course, the same in x', y', z' as in x, y, z.

* This is always possible, since the material body or medium moves relatively to S in a purely *translational* manner.

Now, $\mathbf{r}'$ being the above vector characterising any given point P' of the moving body or medium, the new variable t' is defined by

$$t' = t - \frac{1}{c^2}(\mathbf{r}'\mathbf{v}), \tag{1}$$

and is called the **local time** at P'. Since the scalar product in the second term vanishes for $\mathbf{r}' \perp \mathbf{v}$, the local time coincides with the 'universal' one at all points lying on the plane passing through O' and perpendicular to the direction of motion. But at all other places the new and the old time differ from one another, the local time being behind the 'universal' time in the anterior portion of the body, and the reverse being the case in its posterior portion (Fig. 6). In Cartesians, if $\mathbf{v} = \mathbf{i}v_1 + \mathbf{j}v_2 + \mathbf{k}v_3$, the local time is

$$t' = t - (x'v_1 + y'v_2 + z'v_3)/c^2,$$

or if $\mathbf{i}$ be taken along the direction of motion, $t' = t - x'v/c^2$.

Notice that Lorentz's local time, as just defined, has nothing physical about it. It is merely an auxiliary mathematical quantity to be used instead of the 'universal' time t in order to simplify the form of equations (L). It is constructed expressly for this purpose, and serves it excellently.

In fact, taking instead of $\mathbf{r}'$, t (or x', y', z', t)

$$\mathbf{r}',\ t'$$

as the new independent variables, and denoting the divergence and curl in terms of the new variables by

$$\text{div}' \quad \text{and} \quad \text{curl}',$$

we obtain, for example, by (1) and by the third of equations (L),

$$\text{div}\,\mathbf{M} = \text{div}'\,\mathbf{M} + \frac{1}{c}\mathbf{v}\,\text{curl}\,\mathbf{E}'$$

$$= \text{div}'\,\mathbf{M} - \frac{1}{c}\text{div}\,\mathrm{V}\mathbf{v}\mathbf{E}',$$

since $\text{curl}\,\mathbf{v} = 0$, by hypothesis. But for $\mathrm{V}\mathbf{v}\mathbf{E}'$, as for any vector normal to $\mathbf{v}$, we have, obviously, $\text{div} = \text{div}'$. Hence, by the fifth of (L),

$$\text{div}\,\mathbf{M} = \text{div}'\left(\mathbf{M} - \frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}'\right) = \text{div}'\,\mathbf{M}'.$$

Thus, the fourth of equations (L), $\mathrm{div}\,\mathbf{M}=0$, becomes, in the new variables, $\mathrm{div}'\,\mathbf{M}'=0$. Similarly, the second of (L), $\mathrm{div}\,\mathfrak{E}=0$, is transformed into $\mathrm{div}'\,\mathfrak{E}'=0$, where $\mathfrak{E}'$ is a new vector defined by the formula

$$\mathfrak{E}'=\mathfrak{E}+\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{M}. \tag{2}$$

Using this new vector and the vector $\mathbf{M}'$, defined by the fifth equation, the remaining equations (L) may be transformed, with equal ease, to the new variables.

The result is surprisingly simple. The system of Lorentz's equations (L) for a moving medium takes with the new variables $\mathbf{r}'$, $t'(x', y', z', t')$ the form

$$\left.\begin{array}{ll} \dfrac{\partial\mathfrak{E}'}{\partial t'}=c\,.\,\mathrm{curl}'\,\mathbf{M}'\,; & \mathrm{div}'\,\mathfrak{E}'=0 \\ \dfrac{\partial\mathbf{M}'}{\partial t'}=-c\,.\,\mathrm{curl}'\,\mathbf{E}'\,; & \mathrm{div}'\,\mathbf{M}'=0 \\ \mathfrak{E}'=K\mathbf{E}', & \end{array}\right\} \tag{L'}$$

that is to say, *precisely the same form as for a stationary medium*, (L_0), the only difference being that the electromagnetic vectors $\mathbf{E}$, $\mathfrak{E}$, $\mathbf{M}$ are replaced by their *dashed* correspondents, as are also the independent variables $\mathbf{r}$, t.

This remarkable discovery, made by Lorentz, has played a most important rôle not only in his own theory, but also in the subsequent evolution of ideas concerning electromagnetism and optics. Undoubtedly, it may, to a great extent, be regarded as the germ of modern relativistic tendencies. It will therefore be worth our while to treat this subject at some length, and not only as an historical episode.

The above result may be put into the form of what has been called by Lorentz the **Theorem of corresponding states**:

If we have for a stationary medium or system of bodies any solution (of Maxwell's equations L_0*), in which*

$$\mathbf{E},\ \mathfrak{E},\ \mathbf{M}$$

are certain functions of

$$x, y, z, t,$$

we will obtain a solution for the same system of bodies moving with uniform translation-velocity **v**, *taking for*

$$\mathbf{E}', \mathfrak{E}', \mathbf{M}'$$

exactly the same functions of the variables

$$x', y', z' \text{ and } t' = t - \frac{1}{c^2}(\mathbf{vr}').$$

In other words, and somewhat more shortly:

For each state in which **E**, $\mathfrak{E}$, **M** depend in a certain way on x, y, z, t in the stationary system, there is a **corresponding state** in the moving system characterised by $\mathbf{E}'$, $\mathfrak{E}'$, $\mathbf{M}'$ which depend in the same way on x', y', z', t'.

It will be useful to put here together the scattered definitions of the dashed vectors. These are, by (32), Chap. II.,* by (2) and by the fifth of equations (L),

$$\left.\begin{aligned}\mathbf{E}' &= \mathbf{E} + \frac{1}{c}\,\mathrm{V}\mathbf{vM} \\ \mathfrak{E}' &= \mathfrak{E} + \frac{1}{c}\,\mathrm{V}\mathbf{vM} \\ \mathbf{M}' &= \mathbf{M} - \frac{1}{c}\,\mathrm{V}\mathbf{vE}'.\end{aligned}\right\} \qquad (3)$$

As to the coordinate systems, notice that they are in both cases rigidly attached to the material medium or to the system of bodies in question, x, y, z being fixed together with it in the aether, and x', y', z' sharing its motion through the aether.

The above theorem of corresponding states has, of course, like the equations (L) themselves, the character of a first approximation only, terms of the order of $\beta^2 = v^2/c^2$ having been neglected.

The broad and easy applicability of this beautiful theorem of Lorentz is obvious. It will be enough to quote here a few illustrative examples.

* Remembering that **M** itself is of the first order, so that

$$\frac{1}{c}\,\overline{\mathrm{V}\mathbf{pM}} \doteq \frac{1}{c}\,\overline{\mathrm{V}\mathbf{vM}} = \frac{1}{c}\,\mathrm{V}\mathbf{v}\overline{\mathbf{M}},$$

i.e. in the adopted short notation, $\frac{1}{c}\mathrm{V}\mathbf{vM}$.

If, in the stationary medium or system S of bodies, $\mathbf{E}$, $\mathfrak{E}$, $\mathbf{M}$ are *periodical* functions of t, with period T, then, in the moving system S', the vectors $\mathbf{E}'$, $\mathfrak{E}'$, $\mathbf{M}'$ are periodical functions of the local time t', and consequently, at a point P' fixed in S', also of t, *with the same relative period* T. What Lorentz calls the **relative** period is the period of changes going on at a fixed point of the system S' moving relatively to the aether, *i.e.* for a constant $\mathbf{r}'$, whereas the period of changes taking place at a point fixed in the aether, *i.e.* for a constant $\mathbf{r}$, is called the **absolute** period. Similarly, relative rays are distinguished from absolute rays, and so on. Thus, to luminous vibrations in S of a given absolute period correspond luminous vibrations in S' of the same relative period.

If, in certain regions of the stationary system, $E = 0$, etc., then also $E' = 0$, etc., in the corresponding regions of the moving system. Thus, to darkness corresponds darkness. Also, limitations of *beams* in S and S' correspond to one another. *Luminous rays in* S', *of relative period* T, *are refracted and reflected according to the same laws as rays of (absolute) period* T *in* S. The same is true of the distribution of dark and bright *interference* fringes, and consequently also of the concentration of light in a *focus*, by mirrors or lenses, this being a limiting case of diffraction.

But, although the lateral limitations of beams for corresponding states are the same, corresponding *wave normals* in S, S' have generally *different directions*, this being again an immediate consequence of the theorem of corresponding states. In fact, if we have in S, say, plane waves whose normal is given by the unit vector $\mathbf{n}$ and whose velocity of propagation is $\mathfrak{v}$, *i.e.* if $\mathbf{E}$, $\mathfrak{E}$, $\mathbf{M}$ are proportional to a function of the argument

$$(\mathbf{rn}) - \mathfrak{v}t,$$

then, in the moving system, $\mathbf{E}'$, etc., will be the same functions of the argument

$$(\mathbf{r}'\mathbf{n}) - \mathfrak{v}t' = (\mathbf{r}'\mathbf{n}) + \frac{\mathfrak{v}}{c^2}(\mathbf{r}'\mathbf{v}) - \mathfrak{v}t. \tag{4}$$

Consequently, the direction of the wave normal in the moving system will be given by that of the vector

$$\mathbf{N}' = N\mathbf{n}' = \mathbf{n} + \frac{\mathfrak{v}}{c^2}\mathbf{v}. \tag{5}$$

Thus, unless $\mathbf{n} \parallel \mathbf{v}$, the directions of the wave normals in S and S' are different. To state the same thing in Cartesians, the direction-cosines of the wave normal in the moving system will be given by the proportions

$$n_1' : n_2' : n_3' = \left(n_1 + \frac{\mathfrak{v}}{c^2} v_1\right) : \left(n_2 + \frac{\mathfrak{v}}{c^2} v_2\right) : \left(n_3 + \frac{\mathfrak{v}}{c^2} v_3\right).$$

In particular, for a vacuum or, very approximately, for air, in which case $\mathfrak{v} = c$,

$$\mathbf{N}' = \mathbf{n} + \frac{1}{c} \mathbf{v}, \tag{5a}$$

or, in clumsy Cartesians,

$$n_1' : n_2' : n_3' = \left(n_1 + \frac{v_1}{c}\right) : \left(n_2 + \frac{v_2}{c}\right) : \left(n_3 + \frac{v_3}{c}\right).$$

These formulae may, after a slight transformation, be applied at once to the case of astronomical aberration, the relative period being here that reduced according to Doppler's law. Thus Lorentz obtains immediately the right results for air- and water-telescope aberration. (Cf. *Essay*, p. 89.)

To obtain the dragging coefficient it is enough to write the argument (4)

$$(\mathbf{r}'\mathbf{N}') - \mathfrak{v}t = N' \left\{ (\mathbf{r}'\mathbf{n}') - \frac{\mathfrak{v}}{N'} t \right\}.$$

Since here $\mathbf{n}'$ is a unit vector, the velocity of propagation in S' is

$$\mathfrak{v}' = \frac{\mathfrak{v}}{N'} = \mathfrak{v} \left\{ 1 + \frac{\mathfrak{v}^2}{c^2} \beta^2 + \frac{2\mathfrak{v}}{c^2} (\mathbf{vn}) \right\}^{-\frac{1}{2}},$$

or, neglecting the term containing $\beta^2 = (v/c)^2$, developing the square root and neglecting again the second and higher powers of $(\mathbf{vn})/c$,

$$\mathfrak{v}' = \mathfrak{v} - \left(\frac{\mathfrak{v}}{c}\right)^2 (\mathbf{vn}). \tag{6}$$

In particular, if the propagation is in the direction of motion or against it, as in Fizeau's experiment,

$$\mathfrak{v}' = \mathfrak{v} \mp \left(\frac{\mathfrak{v}}{c}\right)^2 v.$$

Thus, the velocity of propagation relative to the aether will be

$$\mathfrak{v} \pm \left\{1 - \left(\frac{\mathfrak{v}}{c}\right)^2\right\} v,$$

and the value of the dragging coefficient

$$\kappa = 1 - \left(\frac{\mathfrak{v}^2}{c^2}\right) = 1 - \frac{1}{\nu^2}.$$

Here $\nu = c/\mathfrak{v}$ is the refractive index of the medium, say water, corresponding to the *relative* period which is connected with the period T of the emitted light by the formula

$$T_{\text{rel}} = \left(1 \pm \frac{v}{\mathfrak{v}'}\right) T \doteqdot \left(1 \pm \frac{v}{\mathfrak{v}}\right) T,$$

second order terms being neglected. Thus, if n be the refractive index for the period T,

$$\nu = n \pm \frac{v}{\mathfrak{v}} T \frac{\partial n}{\partial T},$$

whence Lorentz's formula for the dragging coefficient,

$$\kappa = 1 - \frac{1}{n^2} - \frac{1}{n} T \frac{\partial n}{\partial T},$$

closely agreeing with experiment, as already mentioned in Chapter II.

For purely *terrestrial* experiments, in which not only the observer but also every part of his apparatus and the source of light are attached to the earth, the theorem of corresponding states leads to the following result:

> *The earth's motion has* **no first order** *influence whatever on any of such experiments.*

The possibility of a second order influence remains, of course, in this stage of the research, an open question. For, as will be remembered, before arriving at the macroscopic equations (L), from which the theorem of corresponding states has been seen to follow, β^2-terms have been throughout neglected. In other words, that beautiful theorem, developed and illustrated by a series of most important examples in the fifth section of Lorentz's classical *Essay*, is but a *first order approximation*.

So far everything is quite satisfactory. But now, in the sixth, and last, section of Lorentz's *Essay* the difficulties begin.* In this section Lorentz investigates three problems, of which two concern the rotation of the plane of polarization and Fizeau's polarization experiments. But without dwelling on these, we shall pass straight on to the third one, namely to the famous *interference experiment of Michelson and Morley.* This second order or β^2-experiment, originally suggested by Maxwell,† was performed by Michelson in 1881, and six years later repeated on a larger scale and with a higher degree of exactness by Michelson and Morley.‡ A beam of luminous rays coming from the source *s*, after having been made parallel in the usual way, is divided by the semi-transparent

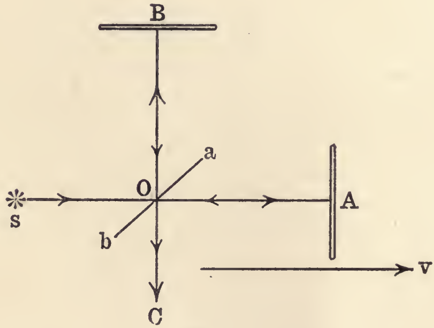


FIG. 7.

plane mirror (half-silvered plate) *ab*, which is inclined at an angle of 45° to sOA, into a transmitted beam OA, and a reflected one OB. After having been reflected by the mirrors placed at A and B (at right angles to OA, OB, which directions are perpendicular to each other), the two beams of light return to the central mirror; here a part of the first beam is reflected along OC and a part of the second

* As is explicitly stated in the title: 'Abschnitt VI.—Versuche, deren Ergebnisse sich nicht ohne Weiteres erklären lassen.'

† See Note at the end of chapter.

‡ A. A. Michelson, 'The relative motion of the earth and the luminiferous ether,' *Amer. Journ. of Science*, 3rd Ser. Vol. XXII., 1881. A. A. Michelson and E. W. Morley, *Sill. Journ.*, 2nd Ser. Vol. XXXI., 1886; *Amer. Journ. of Science*, 3rd Ser. Vol. XXXIV., 1887; *Phil. Mag.*, 5th Ser. Vol. XXIV., 1887. What is given above is but the usual rough scheme; details of the actual arrangement will be found in the original papers quoted and, to a certain extent, also in Michelson's popular book on *Light Waves and their Uses*, where a diagram of the actual apparatus is given (Fig. 108).

beam is transmitted towards *C*, thus producing with one another a system of bright and dark interference fringes, which can be observed through a telescope placed on the line *OC*. To resume it shortly, the paths, taken relatively to the earth, of the two interfering beams of light are:

$$sOAAOC \quad \text{and} \quad sOBBOC.$$

Let *OA* (Fig. 7) be in the direction of the motion of the earth, and consequently also of the apparatus, source and all, with respect to the aether of Fresnel and Lorentz, and let v be the velocity of this motion, *i.e.* the resultant of the earth's orbital velocity, at the time being, and of the velocity of the solar system with respect to the 'fixed stars' or to those 'fixed' stars relatively to which the aether is supposed to be at rest. (Cf. **Note 2**.) On this assumption let us calculate the times taken by the two beams in travelling along their paths. Since the parts *sO* and *OC* are common to both, we have only to consider the intervals of time, say T_1 and T_2, taken to traverse

$$OAAO \quad \text{and} \quad OBBO$$

respectively, where the letters denote, of course, points attached to the apparatus.

Now, as has been already said in Chapter II., in connexion with Maxwell's equations for the 'free aether,' the velocity of light with respect to the aether is always equal $c = 3 \cdot 10^{10}$ cm. sec.$^{-1}$, quite independently of the motion of its source. This is no novel idea at all; Fresnel himself considers it apparently as an obvious matter, when he says (in an early part of his letter, already mentioned) without any further explanations: 'car la vitesse avec laquelle se propagent les ondes est indépendante du mouvement du corps dont elles émanent.' Thus, according to both the classical and the more recent adherents of the aether, *the velocity of light relative to the aether does not depend on the source's motion*; and on the wave-theory there is no reason why it should. Newton's corpuscular theory, revived in a more elaborate form in the writings of the late Dr. Ritz, need not detain us here.

Thus, the mirror *A*, receding from the waves on the part *OA* of their journey, and the mirror *O* moving toward them on their return from *A* to *O*, we have

$$T_1 = \overline{OA}_l \left\{ \frac{1}{c-v} + \frac{1}{c+v} \right\} = \frac{2c}{c^2 - v^2} OA_l,$$

where the index $_l$ is to remind us that OA is 'longitudinal,' *i.e.* along the direction of motion. Putting $v/c=\beta$ and

$$\gamma=\frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma^2 = \frac{c^2}{c^2-v^2} \Rightarrow > 1 \tag{7}$$

we may write shortly, without yet making any use of the smallness of β^2,

$$T_1=\frac{2}{c}\gamma^2\overline{OA}_l. \tag{8}$$

To obtain T_2, the time for the second beam, we could say simply, after the manner of some authors, that the relative velocity of light, being the vector sum of the velocity c parallel to OB and of the velocity v of the aether with respect to the apparatus, perpendicular to OB and directed backwards, is equal $(c^2-v^2)^{\frac{1}{2}}$, so that

$$T_2=2\overline{OB}_t(c^2-v^2)^{-\frac{1}{2}}$$

or

$$T_2=\frac{2}{c}\gamma\overline{OB}_t, \tag{9}$$

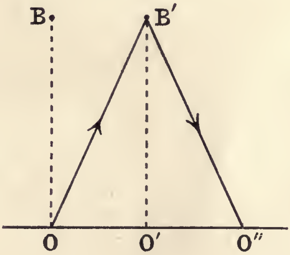


FIG. 8.

where the index $_t$ is to remind us that OB is 'transversal' or perpendicular to the direction of motion. But since this may not seem very satisfactory, we can support it by the following, equally frequent, reasoning which is but formally different from the above short statement. Contemplate for a moment Fig. 8, the paper on which it is drawn being now supposed to be stationary in the aether, and the apparatus moving past it from left to right. Let the centre of the inclined mirror be at O at the instant $t=0$, when the light leaves it, and at O'' at the instant $t=T_2$, when the light returns to it; let B' be the position of B when the beam reaches it, and let O' be the simultaneous position of O. If it be granted

that the three distinct points of the aether, O, O', O'', are the consecutive positions of exactly the same point of the inclined mirror, that is to say, that the ray in question returns to exactly, or sensibly, the same point of the mirror from which it started, then $OB'O''$ will be an isosceles* triangle, so that $OB' = \frac{1}{2}cT_2$, and

$$\tfrac{1}{4}c^2T_2^2 = \tfrac{1}{4}v^2T_2^2 + \overline{OB}_t^2.$$

This gives $T_2 = 2\overline{OB}_t(c^2 - v^2)^{-\frac{1}{2}}$, which is identical with (9).

By (8) and (9) we get for the time-difference of the two beams, by which the phenomenon of their interference is determined,

$$T_1 - T_2 = \frac{2}{c}\gamma\{\gamma\overline{OA}_l - \overline{OB}_t\}. \tag{10}$$

Let us now turn round the whole apparatus through 90°, so that OA becomes transversal, and OB longitudinal. Then we shall have, using dashes to distinguish this case from the above one,

$$T_1' = \frac{2}{c}\gamma\overline{OA}_t, \quad T_2' = \frac{2}{c}\gamma^2\overline{OB}_l,$$

so that the time-difference of the two beams will become

$$T_1' - T_2' = \frac{2}{c}\gamma\{\overline{OA}_t - \gamma\overline{OB}_l\}. \tag{10'}$$

If therefore the fixed-aether theory is true, such a rotation of the apparatus should produce a shift in the position of the interference fringes, corresponding to the change of the time-difference of the two beams, $\Delta = (10) - (10')$, *i.e.*

$$\Delta = \frac{2}{c}\gamma\{\gamma(\overline{OA}_l + \overline{OB}_l) - (\overline{OA}_t + \overline{OB}_t)\}. \tag{11}$$

The indices $_l$ and $_t$, distinguishing between longitudinal and transversal orientation, have been introduced here (contrary to the historical order) only for the sake of subsequent discussions. To Michelson and Morley there was no question of distinguishing between the lengths of a segment in different orientations. To put

*That the above assumption is satisfied with a sufficient degree of accuracy may be seen from **Note 3** at the end of the chapter, where the corresponding Huygens construction is worked out.

ourselves into agreement with their manner of treatment we have, therefore, to write simply

$$\overline{OA}_l = \overline{OA}_t = \overline{OA},$$

$$\overline{OB}_l = \overline{OB}_t = \overline{OB}.$$

To secure these equalities Michelson and Morley mounted the mirrors* and, in fact, the whole of the apparatus, on a heavy slab of stone mounted on a disc of wood which floated in a tank of mercury, so as to be able 'to rotate the apparatus without introducing strains.' In a word, they made the configuration of O, A, etc., 'rigid,' that is to say as rigid as a stone is. On this understanding, formula (11) may be written

$$\Delta = \frac{2}{c}\gamma(\gamma - 1) \, . \, (\overline{OA} + \overline{OB}). \tag{12}$$

As to the mutual relation of $\overline{OA}$, $\overline{OB}$, they were made 'nearly equal,' to suit the well-known requirements for producing neat interference fringes, in each of the two orientations of the apparatus. Moreover, since these lengths or distances enter in the formula only by their *sum*, their equality or non-equality is of no essential importance. We may therefore, without any more ado, write $\overline{OA} = \overline{OB} = L$ or else call the sum of these lengths $2L$. Then, as regards the factor depending on the velocity of motion, we have, by (7),

$$\gamma(\gamma - 1) = (1 - \beta^2)^{-1} - (1 - \beta^2)^{-\frac{1}{2}},$$

or, up to quantities of the second order, *i.e.* neglecting β^4-terms, etc.,

$$1 + \beta^2 - (1 + \tfrac{1}{2}\beta^2) = \tfrac{1}{2}\beta^2.$$

Thus, the second-order effect to be expected on the stationary-aether theory would be determined by the change of the time-difference of the two beams

$$\Delta = \frac{2\beta^2}{c} L. \tag{12a}$$

If T be the period of the light and $\lambda = cT$ the wave-length, the corresponding shift $s = \Delta / T$ of the interference bands, measured as a fractional part of the distance of two neighbouring bands, would be given by

$$s = \beta^2 \frac{2L}{\lambda}. \tag{13}$$

* In the actual experiment not three but sixteen in number.

The length $2L$, which in Michelson's original apparatus was too small, was in Michelson and Morley's experiment (1887) increased to about 22 metres, by multiple reflection from suitably placed mirrors. And since, for sodium light, $\lambda = 5{\cdot}89 \,.\, 10^{-5}$ cm., the ratio $2L/\lambda$ had nearly the value $\cdot 37 \,.\, 10^{8}$. As regards β^2, we should have, taking for v simply the earth's orbital velocity, *i.e.* 30 kilom. per second, $\beta^2 = 10^{-8}$. It is true that, at least in some of the experiments, the rays of light, being horizontal, made a considerable angle with the earth's orbit, but on the other hand the motion of the whole solar system exerted a favourable influence, so as to double the value of β^2 (as was already mentioned). So that to put β^2 equal 10^{-8} is certainly not to overestimate its value considerably. Thus the shift should be on the stationary-aether theory, in round figures,

$$s = 0{\cdot}4 \text{ of a fringe width.}$$

In no case, however, did the actual displacement of the fringes exceed $\cdot 02$, and probably it was less than $\cdot 01$, *i.e.* less than $\frac{1}{40}$th of the expected value. The experiment was repeated in 1905 by Morley and Miller* with considerably increased accuracy, and their result was that, if there is any fringe-shift of the kind expected, it is something like $s = \cdot 0076$ instead of $1{\cdot}5$, *i.e.* not greater than one two-hundredth of the computed value.†

Thus, not nearly the expected second-order effect of the earth's motion relatively to the aether was observed. It seems, therefore, reasonable to say at least that, as far as we know, the above Δ is *nil.*

In order to explain this negative result and to save, at the same time, the stationary-aether theory, Lorentz has had recourse to a peculiar hypothesis, constructed *ad hoc*, which occurred to him independently of Fitzgerald, who was the first to suggest it.‡ It is

* E. W. Morley and D. C. Miller, *Phil. Mag.*, Vol. VIII. p. 753, 1904; *Phil. Mag.*, Vol. IX. p. 680, 1905.

† As to various objections raised against the correctness of the interference experiment by Sutherland, Lüroth and Kohl, and their refutation by Lodge, Lorentz, Debye and Laue, see the 'Literaturübersicht' in J. Laub's report 'Ueber die experimentellen Grundlagen des Relativitätsprinzips,' *Jahrbuch der Radioaktivität und Elektronik*, Vol. VII. p. 405, 1910.

‡ Cf. Lorentz's *Essay*, p. 122 (1895), where reference is made to a paper of his, dated 1892-93. As regards Fitzgerald, we read in *The Ether of Space* by Sir Oliver Lodge (London, 1909, p. 65), referring to that hypothesis: 'It

now widely known under the name of **the contraction hypothesis**, and it consists in assuming that, in Lorentz's words, 'the dimensions of a solid body undergo slight changes, of the order β^2, when it moves through the ether,' namely a longitudinal contraction amounting to $\frac{1}{2}\beta^2$ per unit length or, more generally, both a transversal lengthening and a longitudinal ~~length~~shortening, ϵ and δ, per unit length, such that $\epsilon - \delta = \frac{1}{2}\beta^2$. This would amount for the whole earth to about 6·5 centimetres only.

To see at once that the negative result of the Michelson experiment is thus accounted for and to grasp as clearly as possible the nature of the hypothesis, let us return to the more general formula (11) for Δ, from which (12) or (12*a*) followed by identifying $\overline{OA}_l$ with $\overline{OA}_t$, and similarly $\overline{OB}_l$ with $\overline{OB}_t$. Now, to simplify matters, assume $\overline{OB}_l = \overline{OA}_l$ and $\overline{OB}_t = \overline{OA}_t$ (which, as we saw, is of no essential importance), but on the other hand *distinguish between $\overline{OA}_l$ and $\overline{OA}_t$*. Then formula (11), valid by the fixed-aether theory, will become

$$\Delta = \frac{4}{c}\gamma(\gamma\overline{OA}_l - \overline{OA}_t)\,; \tag{14}$$

and since $\Delta = 0$, by experience, we have to write, in order to respect both that theory *and* experience,

$$\overline{OA}_l = \frac{1}{\gamma}\overline{OA}_t = \overline{OA}_t(1-\beta^2)^{\frac{1}{2}},$$

or, up to quantities of the second order,

$$\overline{OA}_l : \overline{OA}_t = 1 - \tfrac{1}{2}\beta^2,$$

which is the Fitzgerald-Lorentz hypothesis.

Notice that it would be a perfectly idle thing to quarrel whether $\overline{OA}_l$ is shortened, while $\overline{OA}_t$ remains unchanged, by the earth's motion through the aether, or whether OA_t alone is lengthened, or, finally, whether both are changed in suitable proportions. The only thing we are required by the aether theory and by experiment to do is to consider the *ratio* of the lengths of one and the same 'material'

was first suggested by the late Professor G. F. Fitzgerald, of Trinity College, Dublin, while sitting in my study at Liverpool and discussing the matter with me. The suggestion bore the impress of truth from the first.' Happy are those who are gifted with that immediate feeling for 'truth.'

segment OA, or shortly L, in those two orientations as being equal to $1-\frac{1}{2}\beta^2$, or, more rigorously,

$$L_l : L_t = \sqrt{1-\beta^2}. \tag{15}$$

This implies that for $\beta=0$, *i.e.* if the earth stopped moving through the aether, or nearly so, we should have $L_l=L_t$, say, both equal to L_0. But it cannot inform us as to the ratio which either length bears to L_0, when the earth *is* moving through that medium; moreover, such considerations are, thus far, physically meaningless.

At any rate, Lorentz soon decided in favour of a purely longitudinal contraction, which amounts to writing

$$L_t=L_0 \quad \text{and} \quad L_l=\frac{L_0}{\gamma}=L_0\sqrt{1-\beta^2}. \tag{15a}$$

In doing so he based himself on certain results obtained from the fundamental (microscopic) equations in an early part of his classical *Essay*, to be mentioned presently. That this, in fact, was his choice we see explicitly from the shape attributed by him to moving electrons. While Abraham's electron is and remains always a sphere, being rigid in the classical sense of the word, Lorentz's electron is a sphere of radius R, say, when at rest, and becomes flattened longitudinally, when in uniform motion, to a rotational ellipsoid of semiaxes

$$\frac{1}{\gamma}R,\ R,\ R.$$

Such an electron, of homogeneous surface- or volume-charge, is now generally known as the *Lorentz electron*. The history of its rivalry with the rigid one, and of its rather victorious issue from the contest, need not detain us here. It is, besides, sufficiently well known.

Lorentz's attitude towards the contraction hypothesis may be seen best from his own words, written in 1909 (*Electron Theory*, p. 196):

'The hypothesis certainly looks rather startling at first sight, but we can scarcely escape from it, so long as we persist in regarding the ether as immovable. We may, I think, even go so far as to say that, on this assumption, Michelson's experiment *proves* the changes of dimension in question, and that the conclusion is no less legitimate than the inferences concerning the dilatation by heat or the changes of the refractive index that have been drawn in many other cases from the observed positions of interference bands.'

The obvious criticism of the above comparison may be left to the reader.

As regards the justification of the contraction hypothesis which to an unprepared mind certainly does 'look rather startling,' Lorentz observes in his original *Essay* of 1895 (p. 124) that we are led precisely to the change of dimensions defined by (15*a*), if, disregarding the molecular motion, we assume that the attractive and repulsive forces acting on any molecule of a solid body which 'is left to itself' are in mutual equilibrium, and if we apply to these molecular forces the same law which, by the fundamental equations, holds for electrostatic actions. It is true, as Lorentz himself confesses, that 'there is, of course, no reason' for making the second of these assumptions. But those who entertain the hope of constructing an electromagnetic theory of matter will easily adhere to it. To obtain the law in question return to the fundamental electronic equations (I.), Chap. II., and introduce the so-called *vector potential* **A** and the *scalar potential* ϕ, satisfying the differential equations

$$\left.\begin{aligned}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi &= \rho \\ \left(\frac{1}{c}\frac{\partial t^2}{\partial^2} - \nabla^2\right)\mathbf{A} &= \frac{1}{c}\rho\mathbf{p}\end{aligned}\right\} \qquad (16)$$

and subject to the condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0. \qquad (17)$$

Then all of the equations (I.) will be satisfied by

$$\left.\begin{aligned}\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi \\ \mathbf{M} &= \operatorname{curl}\mathbf{A},\end{aligned}\right\} \qquad (18)$$

so that every electromagnetic problem is reduced to finding the potentials according to (16) and (17). Suppose, now, that a material body moves as a whole, relatively to the aether or to the system *S*, with uniform translational velocity **v**, and that all the electrons it carries are at rest with respect to the body. Then the above **p** will have throughout the constant value **v**, so that, by (16),

$$\mathbf{A} = \frac{1}{c}\mathbf{v}\phi. \qquad (19)$$

Thus everything is made to depend on ϕ alone. Take the x-axis in S along the direction of motion, so that $\mathbf{v}=v\mathbf{i}$, $\mathbf{A}=\mathbf{i}\beta\phi$, and suppose that the electromagnetic field is invariable with respect to the material body. This assumption will be satisfied if ϕ is supposed to depend only on the coordinates attached to the body,

$$\xi=x-vt, \quad \eta=y, \quad \zeta=z.$$

Thus we shall have

$$\frac{\partial}{\partial t}=-v\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}, \text{ etc.,}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=\beta^2\frac{\partial^2}{\partial \xi^2},$$

and the equation for ϕ will become

$$\frac{1}{\gamma^2}\frac{\partial^2\phi}{\partial \xi^2}+\frac{\partial^2\phi}{\partial \eta^2}+\frac{\partial^2\phi}{\partial \zeta^2}=-\rho, \tag{20}$$

while the condition (17) will be satisfied identically. Here

$$\gamma^{-2}=(1-\beta^2),$$

as above. Again, by (18),

$$\mathbf{E}=-\left(\mathbf{i}\frac{\partial}{\gamma^2}\frac{\partial}{\partial \xi}+\mathbf{j}\frac{\partial}{\partial \eta}+\mathbf{k}\frac{\partial}{\partial \zeta}\right)\phi,$$

$$\mathbf{M}=\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}=\beta\mathrm{V}\mathbf{i}\mathbf{E},$$

whence the ponderomotive force per unit charge, or Lorentz's *electric force*, $\mathbf{E}+\beta\mathrm{V}\mathbf{i}\mathbf{M}$, (10), Chap. II., which we shall now denote by $\mathfrak{E}$ (since the *dashed* $\mathbf{E}$ would be misleading),

$$\mathfrak{E}=-\nabla\left(\frac{\phi}{\gamma^2}\right), \tag{21}$$

where $\nabla=\mathbf{i}\,\partial/\partial\xi+\mathbf{j}\,\partial/\partial\eta+\mathbf{k}\,\partial/\partial\zeta=\mathbf{i}\,\partial/\partial x+\ldots$ is the Hamiltonian (here acting as the slope), taken with respect to the aether or, which in our case is the same thing, with respect to the material body. Thus, the electric force is derived from a scalar potential ϕ/γ^2, precisely as in ordinary electrostatics. By the way, ϕ/γ^2 is called the *convection potential.* Notice that it is $\mathfrak{E}$, the electric force, and not the 'dielectric displacement' $\mathbf{E}$, that has a scalar potential.

Now, supposing always $\beta^2 < 1$ and consequently γ real, write

$$x' = \gamma\xi, \quad y' = \eta, \quad z' = \zeta, \tag{22}$$

and denote the corresponding Hamiltonian, $\mathbf{i}\,\partial/\partial x' + \text{etc.}$, by ∇'. Then (20) will become

$$\nabla'^2\phi = -\rho. \tag{23}$$

To adopt for the moment Lorentz's notation, call the moving material body or system of bodies the system S_1, and compare it with a system S_2 which is *fixed* in the aether and which is obtained from S_1 by stretching all its constituent bodies, together with the electrons, longitudinally in the ratio $\gamma : 1$, so that to any point ξ, η, ζ of S_1 corresponds the point x', y', z' of S_2, and so that corresponding volume-elements, $d\tau$ and $d\tau' = \gamma\, d\tau$, contain *equal charges*. Then, ρ and ρ' being the densities of electric charge at corresponding points,

$$\rho' = \frac{1}{\gamma}\rho,$$

and, by (23),

$$\nabla'^2\phi = -\gamma\rho'.$$

If then ϕ' be the scalar, electrostatic, potential in S_2, so that

$$\nabla'^2\phi' = -\rho',$$

we shall have

$$\phi' = \frac{1}{\gamma}\phi,$$

and consequently, instead of (21), using (22),

$$\mathfrak{F} = -\frac{1}{\gamma}\nabla\phi' = -\mathbf{i}\frac{\partial\phi'}{\partial x'} - \frac{1}{\gamma}\left(\mathbf{j}\frac{\partial\phi'}{\partial y'} + \mathbf{k}\frac{\partial\phi'}{\partial z'}\right).$$

But the electric force in the stationary system S_2 is

$$\mathfrak{F}' = -\nabla'\phi' = -\mathbf{i}\frac{\partial\phi'}{\partial x'} - \mathbf{j}\frac{\partial\phi'}{\partial y'} - \mathbf{k}\frac{\partial\phi'}{\partial z'}.$$

Therefore, using the indices $_l$ and $_t$ to denote the longitudinal and the transversal components of the electric forces,

$$\mathfrak{F}_l = \mathfrak{F}_l'\,; \quad \mathfrak{F}_t = \frac{1}{\gamma}\mathfrak{F}_t' = \mathfrak{F}_t'\sqrt{1-\beta^2}, \tag{24}$$

and since charges of corresponding elements are equal, exactly the same relations will hold between the ponderomotive forces acting on each electron in the moving system S_1 and on the corresponding electron in the stationary system S_2.

This is the 'law' alluded to. Now, suppose that it is obeyed by the molecular forces keeping together the parts of a moving solid which, disregarding its interior molecular and electronic motions, is to be taken for the system S_1. Then, if the molecular forces balance each other in the corresponding stationary body S_2, they will do so in the moving body S_1. But, by (22), S_1 is the body S_2 contracted longitudinally with preservation of its transversal dimensions, exactly as in (15*a*), and the motion would produce this flattening 'by itself.' Whence Lorentz's justification of the contraction hypothesis.

Thus, the longitudinal contraction, though at first manifestly invented *ad hoc*, to account for the negative result of the Michelson experiment, found a kind of legitimate support by being brought into connexion with the fundamental assumptions of the electron theory. But the cure of the disease has not been radical. In fact, the idea naturally suggested itself, that the Lorentz-Fitzgerald contraction, like an ordinary strain, might give rise to double refraction, of the order β^2, in solids or liquids, a property which should be directionally connected with the earth's motion round the sun. But here again the result of experiments has been sensibly negative. Lord Rayleigh's* experiments (1902) with liquids (water and carbon disulphide) as well as those with solids, with glass plates piled together, have given no trace of an effect of the expected kind. At least, if there was any effect on turning round the apparatus, it was less than $\frac{1}{100}$th of that sought for. Rayleigh's experiment was then repeated (1904) by Brace† with considerably increased accuracy, and the result has again been negative: the relative retardation of the rays due to the supposed double refraction should be of the order 10^{-8}, whereas, if existent at all, it was certainly less than $5 \cdot 10^{-11}$, in the case of glass, and even less than $7 \cdot 10^{-13}$, in the case of water.

To account for these obstinately negative results, and with a view to settle the matter once and for ever, Lorentz undertook what he

* Lord Rayleigh, *Phil. Mag.*, Vol. IV. p. 678, 1902.

† D. B. Brace, *Phil. Mag.*, Vol. VII. p. 317, 1904; *Boltzmann-Festschrift*, p. 576, 1907.

thought a radical discussion of the whole subject, that is to say, of the electromagnetic phenomena in a uniformly moving system, not as hitherto for small values of v, but for *any* velocity of translation smaller than that of light, *i.e.* for any $\beta < 1$. Lorentz's ideas, laid down in a paper published in 1904,* are fully developed in his *Columbia University Lectures*, already quoted (p. 196 *et seq.*). His aim was now to reduce, 'at least as far as possible,' the electromagnetic equations for a moving system to the form of those that hold for a system at rest—always, of course, relatively to the aether—without neglecting either β^2- or, in fact, terms of any order whatever.

It will be remembered that even in his first approximation, *i.e.* when neglecting β^2-terms, Lorentz employed the 'local time' $t' = t - (\mathbf{vr})/c^2$, or, measuring x along the line of motion,

$$t' = t - \frac{v}{c^2}x.\dagger \qquad (a)$$

Then the necessity of accounting for the negative result of Michelson's interference experiment brought him to the contraction hypothesis, according to which the longitudinal dimensions of the moving system are reduced in the ratio $1 : \gamma^{-1}$, where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, while the transversal ones remain unchanged. This contraction corresponds to $t = \text{const.}$, and consequently may easily be shown to be equivalent to transforming x, y, z, the coordinates of a point with respect to axes fixed in the aether, or the 'absolute' coordinates, into

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z. \qquad (b)$$

It is true that the transformation (a) was as yet purely formal, and that the contraction, or (b), was introduced by Lorentz first *ad hoc*, but afterwards to be justified. But at anyrate, having already (a) and (b), Lorentz has been naturally led to investigate in a general way the consequences of introducing, instead of x, y, z, t,

* H. A. Lorentz, 'Electromagnetic phenomena in a system moving with any velocity smaller than that of light,' *Proc. Amsterdam Acad.*, Vol. VI. p. 809; 1904.

† Here, according to the original definition of 'local time,' p. 66, we should have rigorously (instead of the coordinate x, measured in the fixed framework) $x - vt$, so that $t' = (1 + \beta^2)t - \frac{v}{c^2}x$. But, since at that stage β^2-terms were neglected, we could write simply x instead of $x - vt$. The symbols x', etc., in what follows are not to be confounded with the x', etc., of page 66.

new independent variables, called by him the **effective** coordinates and the effective time,

$$\left.\begin{aligned} x' &= \lambda\gamma(x - vt), \quad y' = \lambda y, \quad z' = \lambda z, \\ t' &= \lambda\gamma\left(t - \frac{v}{c^2}x\right), \end{aligned}\right\} \qquad (25)$$

where γ is as above and λ is a numerical coefficient of which Lorentz, provisionally, assumes only that it is a function of v alone, whose value equals 1 for $v=0$ and differs from 1 by an amount of the order β^2 for small values of the ratio $\beta = v/c$.* Introducing the new variables (25) into the fundamental electronic equations, (I.), Chap. II., and defining new vectors $\mathbf{E}'$, $\mathbf{M}'$,

$$\left.\begin{aligned} E_1' &= \lambda^{-2}E_1, \quad E_2' = \gamma\lambda^{-2}(E_2 - \beta M_3), \quad E_3' = \gamma\lambda^{-2}(E_3 + \beta M_2), \\ M_1' &= \lambda^{-2}M_1, \quad M_2' = \gamma\lambda^{-2}(M_2 + \beta E_3), \quad M_3' = \gamma\lambda^{-2}(M_3 - \beta E_2), \end{aligned}\right\} \qquad (26)$$

and also, instead of the relative velocity $\mathbf{p} - \mathbf{v}$ of an electric particle, the vector

$$\mathbf{p}' = \gamma\{\mathbf{i}\gamma(p_1 - v_1) + \mathbf{j}(p_2 - v_2) + \mathbf{k}(p_3 - v_3)\},$$

i.e. with the above choice of axes, simply

$$\mathbf{p}' = \gamma\{\mathbf{i}\gamma(p_1 - v) + \mathbf{j}p_2 + \mathbf{k}p_3\}, \qquad (27)$$

and, instead of the density ρ,

$$\rho' = \gamma\lambda^{-3}\rho, \qquad (28)$$

Lorentz obtained again the equations (I.) with dashes,

$$\partial\mathbf{E}'/\partial t' + \rho'\mathbf{p}' = c \,.\, \mathrm{curl}'\,\mathbf{M}', \quad \text{etc.,}$$

but with the difference that $\mathrm{div}\,\mathbf{E} = \rho$ was replaced by

$$\mathrm{div}'\,\mathbf{E}' = \left[1 - \frac{1}{c^2}(\mathbf{v}\mathbf{p}')\right]\rho', \qquad (29)$$

* *Columbia University Lectures*, p. 196. The above v, γ, λ stand for Lorentz's w, k, l respectively. A transformation equivalent to (25) was previously applied by Voigt, as early as 1887, to equations of the form $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = 0$; 'Ueber das Doppler'sche Princip, *Göttinger Nachrichten*, 1887, p. 41. Lorentz himself states (*loc. cit.*, p. 198; 1909) that Voigt's paper had escaped his notice all these years, and adds: 'The idea of the transformation' (25) 'might therefore have been borrowed from Voigt, and the proof that it does not alter the form of the equations for the *free* ether is contained in his paper.'

not by $\mathrm{div}'\,\mathbf{E}' = \rho'$. Thus, the fundamental equations for the free aether ($\rho = \rho' = 0$) turned out to be rigorously invariant with respect to the transformation (25), which, especially for $\lambda = 1$, has since been universally called the **Lorentz transformation.** The same invariance holds also in the general case, that is to say, in the presence of electric charges, but for the slight deviation given by (29).

Using this result, Lorentz generalized his *Theorem of corresponding states* for any velocity v smaller than c, and succeeded in showing that the theorem thus extended not only accounts for the contraction required by the result of the Michelson experiment, but that it explains, among other things, why Lord Rayleigh and Brace failed to detect a double refraction due to the earth's orbital motion. A discussion of the formulae for the longitudinal and transversal masses of an electron, which need not detain us here,* led Lorentz to attribute to the coefficient λ (his l) the value 1, whereby the transformation formulae (25) and (26) were reduced to

$$\left.\begin{aligned} &x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \\ &t' = \gamma\left(t - \frac{v}{c^2}x\right), \end{aligned}\right\} \tag{30}$$

and

$$\left.\begin{aligned} &E_1' = E_1, \quad E_2' = \gamma(E_2 - \beta M_3), \quad E_3' = \gamma(E_3 + \beta M_2), \\ &M_1' = M_1, \quad M_2' = \gamma(M_2 + \beta E_3), \quad M_3' = \gamma(M_3 - \beta E_2). \end{aligned}\right\} \tag{31}$$

With this specialization, Lorentz's modified theory, which in its essence was built up in 1904, satisfied the requirements of self-consistency and accounted for the negative results of all, second as well as first order, terrestrial experiments intended to show our planet's motion through the aether. In other words, by modifying and gradually extending his original theory, Lorentz obtained the desired physical *equivalence* of the 'moving' system S', with its effective coordinates and time x', y', z', t', and of a corresponding 'stationary' system with its absolute coordinates and time x, y, z, t.

But still one of the two systems S, S', namely S, was *privileged*, being regarded by Lorentz as 'fixed in the aether.' Their equivalence, as indicated persistently by such numerous experiments, was not placed as the basis of the theory, but followed as the result of long, laborious, and rather artificial constructions, intended to com-

*See *Columbia University Lectures*, pp. 211-212.

pensate gradually the pretended play of the 'aether.' For, to repeat, Lorentz continued to assume this hypothetical medium of his classical *Essay* in his extended theory, dated 1904, and adheres to it even now, if we may judge from the last sentences of his American Lectures (p. 230). Not only is the aether for Lorentz a unique framework of reference, but he 'cannot but regard it as endowed with a certain degree of substantiality.' According to this standpoint, then, there certainly is such a thing as the aether, though every physical effect of the motion of ordinary, ponderable matter through it, being compensated by more or less intricate processes, remains undiscoverable for ever.

As regards the above transformation of Lorentz, we may further notice here that Poincaré made, in 1906, an extensive use of its more general form (25) [*Rend. del Circolo mat. di Palermo*, Vol. XXI. p. 129] for the treatment of the dynamics of the electron and also of universal gravitation. Some of Poincaré's results continue even now to be of considerable interest.

In the meantime, 1905, Einstein published his paper on 'the electrodynamics of moving bodies,'* which has since become classical, in which, aiming at a perfect reciprocity or equivalence of the above pair of systems, S, S', and denying any claims for primacy to either, he has investigated the whole problem from the bottom. Asking himself questions of such a fundamental nature, as what is to be understood by 'simultaneous' events in a pair of distant places, and dismissing altogether the idea of an aether, and in fact of any unique framework of reference, he has succeeded in giving a plausible support to, and at the same time a striking interpretation of, Lorentz's transformation formulae and the results of Lorentz's extended theory. Einstein's fundamental ideas on physical time and space, opening the way to modern Relativity, will occupy our attention in the next chapter.

* A. Einstein, *Annal. der Physik*, Vol. XVII. p. 891; 1905.

NOTES TO CHAPTER III.

Note 1 (to page 72). It seems desirable to quote here after Lorentz (*Abhandlungen über theor. Physik*, Vol. I. p. 386, footnote) a passage from Maxwell's letter 'On a possible mode of detecting a motion of the solar system through the luminiferous ether,' published after his death in *Proc. Roy. Soc.*, Vol. XXX. (1879-1880), p. 108:

'In the terrestrial methods of determining the velocity of light, the light comes back along the same path again, so that the velocity of the earth with respect to the ether would alter the time of the double passage by a quantity depending on the square of the ratio of the earth's velocity to that of light, and this is quite too small to be observed.'

Note 2 (to page 73). Usually, at least in all text-books, it is simply said: 'Suppose that the aether remains at rest, and let v = the velocity of the apparatus, *i.e.* of the earth in its orbit.' For this to be correct, the aether would have to be at rest with respect to our sun. But when astronomical aberration is in question, we are told that the aether is stationary with respect to the 'fixed stars,' say, with respect to the constellation of Hercules, which, I hope, is 'fixed' enough. Now, as has incidentally been mentioned (p. 17), the sun or the whole solar system has a uniform velocity of something like 25 kilometres per second towards that constellation, which, being nearly equal in absolute value to the earth's orbital velocity (30 klm. per sec.), certainly cannot be neglected. Thus, the velocity (v) of Michelson's interferometer with respect to the aether would oscillate to and fro, in half-year intervals, between considerably distinct maximum- and minimum-values. According to Lorentz ('De l'influence du mouvement de la terre sur les phénomènes lumineux,' 1887, reprinted in *Abhandlungen*, Vol. I.; see p. 388) the resultant of the earth's orbital and the solar system's velocity had at the time when Michelson was performing his experiment both a direction and an absolute value 'very favorable' to the effect sought for, even so much as to double the displacement of the fringes expected. I am not aware whether or no the defenders and the adversaries of the aether have discussed this circumstance with sufficient care. But at any rate it seemed worth noticing here. Of course, it is for the adherents of the aether (and not those of empty space) to tell us explicitly with respect to what celestial bodies, the sun, or Hercules or other groups of stars, the aether is to be stationary, if it be granted that the parts of that medium do not move relatively to each other. For these stars certainly move relatively to one another.

I cannot help remarking here that it is repugnant to me to think of an omnipresent *rigid* aether being once and for ever at rest relatively rather to one star than to another. For, this medium, unlike Stokes's aether, being non-deformable and not acted on by any forces whatever, none of the celestial bodies, be it ever so conspicuous in bulk or mass, can claim for itself this primacy of holding fast the aether. The bare idea

of action exerted upon the aether by material bodies being dismissed at the outset, there is nothing which could confer this distinctive privilege upon any one of them. But, then, I am quite aware that what 'is repugnant to think of' may not necessarily be wrong altogether. There are other reasons to be urged against the aether.

Note 3 (to page 75). Let a plane wave σ (Fig. 9) proceed towards the inclined mirror (half-silvered plate) Oa in the direction of its motion, *i.e.* from left to right. Let sO, sma represent the incident wave normals, limiting a part of the beam of breadth $\overline{Om}=b$, and let OX be the normal to the mirror, so that $\theta=sOX$ is the angle of incidence. Let the wave reach the centre O of the mirror at the instant $t=0$. Let O_1 and a_1 be the positions of the points O and a of the mirror (both taken in the plane

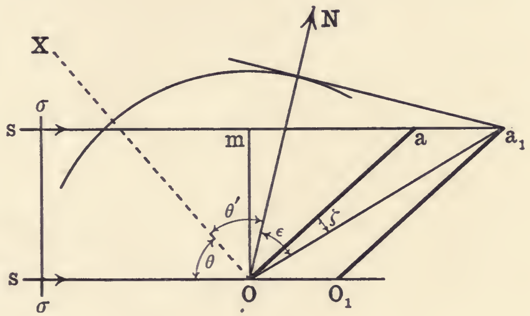


FIG. 9.

of the figure) at a later instant $t=\tau$, when the wave of disturbance reaches a_1, so that

$$\overline{aa_1}=\overline{OO_1}=v\tau.$$

Draw round O a circle with the radius

$$\overline{ma_1}=c\tau\,;$$

then the tangent to this circle, drawn from a_1, will represent the reflected wave, and ON will be the reflected wave normal. To obtain the angle of reflection, $\theta'=XON$, consider the triangles ONa_1 and Oma_1, having the side Oa_1 in common and right angles at m and at N. Since, moreover, their sides ON and a_1m are equal to one another, $\overline{a_1N}=Om=b$, so that the breadth of the beam remains unchanged by reflection, as for a stationary mirror, and

$$\sphericalangle NOa_1\equiv\epsilon=\sphericalangle ma_1O=\frac{\pi}{2}-\theta-\zeta,$$

where $\zeta=\sphericalangle aOa_1$. But $\theta'=\pi/2-\epsilon+\zeta$. Thus, the angle of reflection θ' and the angle of incidence θ are connected by the relation

$$\theta'=\theta+2\zeta, \tag{A}$$

where the angle ζ is determined by the given properties of the parallelogram Oaa_1O_1. Writing

$$\overline{Oa}=\overline{O_1a_1}=l,$$

we have at once

$$\overline{Oa_1}^2=(v\tau)^2+l^2+2v\tau l\,.\sin\theta$$

and

$$v\tau:\overline{Oa_1}=\sin\zeta:\cos\theta;$$

whence

$$\frac{\cos^2\theta}{\sin^2\zeta}=1+\frac{l^2}{(v\tau)^2}+\frac{2l}{v\tau}\sin\theta.$$

But $v\tau=vl\sin\theta/(c-v)$, or $l/v\tau=\dfrac{1-\beta}{\beta\sin\theta}$, so that the required formula for ζ is

$$2\sin\zeta=\frac{\beta\sin(2\theta)}{\sqrt{1-\beta(2-\beta)\cos^2\theta}}. \tag{B}$$

(A) and (B) contain the rigorous solution of the problem, based, of course, on the assumption of a stationary aether.

In Michelson and Morley's experiment, as treated above (Fig. 8), $2\theta=90°$, so that (B) becomes

$$2\sin\zeta=\beta(1-\beta+\tfrac{1}{2}\beta^2)^{-\frac{1}{2}}. \tag{B$_1$}$$

To connect Fig. 9 with Fig. 8, notice that, according to (A), the angle BOB' should be equal to 2ζ. The approximate treatment given in connexion with Fig. 8 (p. 74) amounts to writing

$$\sin(BOB')=v:c=\beta. \tag{C}$$

Now, developing (B$_1$) and remembering that β is a small fraction, we have, up to quantities of the second order,

$$2\sin\zeta=\beta+\tfrac{1}{2}\beta^2,$$

or, neglecting the third and higher powers of the small angle ζ,

$$\sin(2\zeta)=\beta+\tfrac{1}{2}\beta^2.$$

But the term $\frac{1}{2}\beta^2$ appearing in this formula for the angle would give in the final formula for T_2 only terms of the order of β^3 and β^4. Thus, aiming at results which are correct only up to quantities of the second order, we may write the last formula

$$\sin(2\zeta)=\beta,$$

in agreement with (C). Our Huygens-construction shows then that the treatment adopted on page 74 is sufficiently correct for the purpose in question.

That treatment, which is given in all text-books (including also such valuable modern works as Laue's *Relativitätsprinzip*, 1913) without any further remark, would be rigorously correct if O were, say, a point

source of (spherical) waves spreading out in all directions, and not, as it actually is, one of the points of a mirror at which reflection of plane waves is taking place.

A different way of treating rigorously the above question will be found in Lorentz's paper entitled 'De l'influence du mouvement de la terre sur les phénomènes lumineux,' *Arch. néerl.*, Vol. XXI. (1887), pp. 169-172 (reprinted in *Abhandlungen über theor. Physik*, Vol. I. pp. 389-392) and partly also in his *Columbia University Lectures*, p. 194.

The discussion of our general formulae (A), (B) connecting the angle of reflection with that of incidence, for large values of β, may be left to the reader as a curious exercise.

CHAPTER IV.

EINSTEIN'S DEFINITION OF SIMULTANEITY. THE PRINCIPLES OF RELATIVITY AND OF CONSTANT LIGHT-VELOCITY. THE LORENTZ TRANSFORMATION.

We are now sufficiently prepared to grasp the meaning of Einstein's ideas* and to appreciate their relation to the work of his predecessors, especially of Lorentz.

In Chapter I. we have seen how it is possible to define the time as a physically measurable quantity fulfilling certain reasonable and fairly general requirements. Practically, it was the variable t measured by the rotating earth as time-keeper or what, with a slight correction connected with tidal friction, has been called the 'kinetic time.' It has certainly not escaped the reader's notice that the requirements on which that choice was based had nothing absolute or necessary about them, being merely recommended by their simplicity and convenience. But this circumstance need not detain us here any further. Suppose we have secured a clock indicating, with a sufficient degree of precision, the kinetic time t. Suppose we keep that clock at a certain place a, relatively to a given space-framework of reference, say in a certain physical laboratory or astronomical observatory. Thus far we have tacitly assumed that the time t, measured by such a chronometer, is universal, if I may say so, *i.e.* that it is valid for all points of space, for all parts of any system, be it near to our clock or very far from it, be it at rest or moving with respect to it. It is very likely that nobody has ever

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* As laid down in his paper of 1905, already quoted, and then (1907) developed by him more fully in a paper, 'Ueber das Relativitätsprinzip und die aus demselben gezogenen Folgerungen,' *Jahrbuch der Radioaktivität und Elektronik*, Vol. IV. p. 411. In what follows we shall refer principally to the former of these papers by quoting simply the original numbers of its pages.

asserted explicitly this universality and uniqueness of time, but everybody has certainly given to it his tacit consent, and would willingly endorse it if asked to do so. As far as we know, the first to question this pretended universality of time was Einstein.

Our clock, placed at a, indicates the time t, *i.e.* marks different time-instants and measures the intervals between them, to begin with, *only* at the place a, or nearly so. It is, to give it a short name, the time t_a. Suppose that some well-marked instant is chosen as the initial instant, $t_a = 0$. Then, if any event is happening at a or near a, we give to it that date or, as it were, label it with that number t_a which is simultaneously shown by the index of our clock. We are exempted from defining what 'simultaneous' (as well as 'earlier' or 'later') means when applied to a pair of events occurring at the same place or near that place, as the passage of the index through a given division of the dial of our clock and the production of an electric spark closely to it.* But we do not know, beforehand, what we are to understand by saying that of two events occurring at places a, b distant from one another the first occurs earlier or later than the second, or that both are simultaneous. The meaning of these words has to be defined. If the labelling of all possible kinds of events, occurring at distant points, fixed or moving relatively to one another, is to be of any use at all, we must establish the rules according to which we are going to label them with the t-numbers. And first of all we have to decide which of these events have to receive the same labels, *i.e.* we have to define **simultaneity at distant points.**

This notion is to be defined in terms of simultaneity at the same place, which alone is assumed to be known to us, and of some other things or processes which are actually realizable. In other words, distant simultaneity has to be reduced to local simultaneity by some physical process. Abstractly speaking, the choice of such a process is arbitrary, in very wide limits at least; but practically the choice will be reduced to such processes as are of possibly universal occurrence, and which are independent of the capricious peculiarities of different sorts of matter. Einstein has chosen for this purpose the propagation of light *in vacuo.* Gravitation being, chiefly due

* We need not stop here to consider such apparatus as Siemens' 'spark-chronometer,' in which the visible marks corresponding to pairs of events are brought very close to one another, and which enable the modern physicist to fix with a high degree of precision their time-relations.

to its alleged instantaneous action, out of question, this has been, in fact, the only possible choice. Moreover, it was not unprecedented in the history of physics and astronomy, and it suggested itself most obviously because the recent difficulties met with lay in the optical and, more generally, electromagnetic departments of physics.

To an unbiassed mind the question may present itself: Why label everything in the world with t-numbers at all? Such a question is not altogether unreasonable, and it may deserve some careful attention. But once we decide to attach a time-label to every event, we are forced to reduce in some kind of way distant simultaneity to local simultaneity (for pairs of points at rest or moving relatively to one another), and not to delude ourselves with thinking that we know what 'universal simultaneity' means, or that it is, in fact, a self-consistent notion. To have initiated a critical analysis of the concept of simultaneity at all is certainly a great merit of Einstein's.

But let us leave aside these generalities and pass to the definition in question. We shall have to consider in the first place the simpler case of distant points a, b, etc., in relative rest, and then the somewhat intricate case of distant points belonging to systems which are uniformly moving with respect to one another.

Let a, b, etc., be points or 'places' fixed relatively to one another and with respect to a certain space-framework or system S, say, the system of the fixed stars.* Suppose we succeeded in manufacturing at the place a a number of equal clocks, each measuring the same, say the 'kinetic,' time t and set equally or synchronously, and that retaining one of them at a we sent the others to b, etc., together with an equal number of observers who are to remain at those distant places with their clocks for ever. Then, to begin with, we should have as many 'times' as there are places in consideration, t_a, t_b, etc., valid, respectively, for the places a, b, etc., and for their nearest neighbourhoods. For, though all of these clocks were manufactured equally at a, we do not know whether they continue to be 'equal' or permanently synchronous, when one of them is

*In his paper (p. 892) Einstein begins with taking, for the purpose of his definition of simultaneity, that 'system of coordinates in which Newton's mechanical equations are valid.' But it seems advisable not to appeal at the outset, and in connexion with such a fundamental definition, to Newtonian mechanics, especially as it requires, according to the relativistic view itself, some essential, though numerically slight, modifications. On the other hand, the physical specification of what has been called above *the system S* will appear presently without recourse to any theory of mechanics.

still kept at a, while the others are sent far away, to the places b, etc. More than this, we do not know what their being synchronous or not, when far apart, means. We have yet to fix how we are going to test it. To invoke the preservation of rate of clocks of 'good make' in spite of their being carried to distant places, on the title of the high precision of their mechanisms, would not help us out of the difficulty. For, supposing we also decided to assert such infallible and rigorous permanence, at different places within S, of the mechanical laws, necessarily involved, still we should have to verify whether the accessorial conditions of validity of those laws (and practically there would be a host of such conditions) are fulfilled at and round each place in question. To avoid this verification, which soon would prove to be a difficult task, we must have some means of *testing* in a direct manner the synchronism of our distant clocks and, more generally, of correlating with one another the times t_a, t_b, etc., without being obliged to enter upon the properties and structure of the corresponding clock mechanisms.*

Now, the kind of test adopted by Einstein, and constituting at the same time the essence of his definition of distant simultaneity, is as follows.

Let an observer stationed at a send a flash of light at the instant t_a (as indicated by the a-clock) towards b, where it arrives at the instant t_b (according to the b-clock). Let another observer send it back from b without any delay, or let the flash be automatically reflected at b, towards a, where it returns at the instant t_a'. Then the b-clock is said, by definition, to be **synchronous** with the a-clock, if

$$t_a' - t_b = t_b - t_a. \tag{1}$$

This amounts to requiring, *per definitionem*, that 'the time' employed by light to pass from a to b should be *equal* to 'the time' employed to return from b to a. Instead of (1) we may write, equivalently,

$$t_b = t_a + \tfrac{1}{2}(t_a' - t_a) = \tfrac{1}{2}(t_a + t_a'). \tag{1a}$$

Thus, the instant of arrival at b is expressed by the arithmetic mean of the a-times of departure and return of the light-signal. Such

* We may notice in this connexion that Einstein's specification (p. 893): 'eine Uhr [at b] von genau derselben Beschaffenheit wie die in A [a] befindliche' is unnecessary and, to a certain extent, misleading.

being the connexion of the a-time and of the b-time, the clock placed at b is said to be synchronous with that placed at a.

This definition of synchronism is supposed to be self-consistent, for any number of clocks placed at different points of the system S, say, besides a and b, at c, d, e, etc. To secure this consistency, Einstein makes, explicitly, the following two assumptions:

1. If the clock at b is synchronous with that at a, then also the clock at a is synchronous with that at b. In other words: clock-synchronism is *reciprocal*, for any pair of places taken in S.

2. If two clocks, placed at a and b, are synchronous with a third clock, placed at c, they are also synchronous with one another. Or, more shortly, clock-synchronism is *transitive* throughout the system S.

This is the way that Einstein himself puts the matter. But it may easily be shown that the first of his assumptions will be fulfilled if we require that 'the time' employed by the light-signal to pass from a to b is *always the same*. In fact, let us denote the a-time, taken generally, by a instead of t_a, and similarly, let us write b instead of the general variable t_b, and let us use the suffixes $_{d,\ a,\ r}$ to denote the instants of departure, arrival and return. Then, if the b-clock is synchronous with the a-clock, we have, by definition, $b_a = \frac{1}{2}(a_d + a_r)$, or

$$b_a - a_d = a_r - b_a = a_a - b_a,$$

for the 'return' at a may be equally well considered as an arrival at that place. Now, if at the instant a_a the flash be sent again towards b, where it arrives at the instant b_r, we have, by our above requirement,

$$b_a - a_d = b_r - a_a,$$

and, by the last equation,

$$a_a - b_a = b_r - a_a.$$

But here b_a is identical with the instant of departure b_d, and, consequently,

$$a_a = \tfrac{1}{2}(b_d + b_r),$$

i.e. the clock placed at a is synchronous with that placed at b. Q.E.D.

A similar treatment of assumption 2. may be left to the reader, who will find sufficient hints in Fig. 10. This assumption will be easily seen to imply that if a pair of flashes be sent out simultaneously

from a, one *via* b, c and the other *via* c, b, they will both return *simultaneously* at a. More generally, the time elapsing between the instant of departure and that of return of the light-signal sent round *abca* will be equal to the time elapsing between departure and return of the signal sent round *acba*, and similarly for every other closed path in S, both times being measured by the clock placed at a. This form of the property attributed to the system S is worthy of being especially insisted upon, as it implies only operations to be performed at one and the same spot. To state this property of the system S, the observer has not to move from his place.

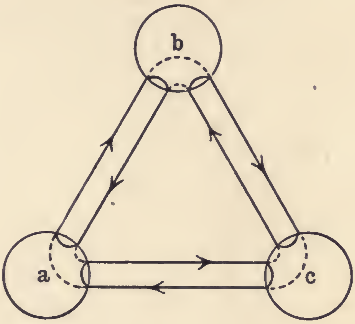

FIG. 10.

Such then are the physical properties of this system of reference.

It is strange that Einstein, after having made explicitly the above assumptions 1. and 2., considers it necessary to add (p. 894) that 'according to experience' the quantity

$$2\frac{\overline{ab}}{a_r - a_d} = c,$$

or, in the notation of formula (1), the quantity

$$2\frac{\overline{ab}}{t_a' - t_a} = c, \tag{2}$$

is to be taken as 'an universal constant (the velocity of light in empty space).' At any rate, if the last assumption is made, for any pair of points a, b in S, once and for ever, then the above statements 1. and 2. are certainly superfluous. But considerations of this order need not detain us here any more.

The properties ascribed to the system S* may be briefly summarised by saying that

> Isotropy and homogeneity with respect to light propagation are postulated throughout S, once and for ever.

In this way the various times, t_a, t_b, etc., originally foreign to one another, are all connected so as to constitute one time only, valid for the whole system, which we may denote simply by t, calling it shortly **the S-time.**

There is, thus far, nothing essentially new in Einstein's procedure. It was more or less unconsciously applied since people began to measure the velocity of light, and even sound, nay, since they began to exchange with one another letters or messages of any kind. The novelty does not come in until the next stage, when the time-labelling is extended to different systems moving (uniformly) with respect to one another.

Let S be as above, and let us consider other systems of reference, S', S'', and so on, each having with respect to S a motion of *uniform* (*rectilinear*) *translation.* Having settled the matter for the system S, *i.e.* having established the S-time, t, let us similarly establish an S'-time, t', an S''-time, t'', etc., and let us see how the times t', t'', etc., are connected with the time t valid for S. It can reasonably be expected that these processes of (time-) labelling of events happening at different places, being undertaken from different standpoints, S, S', S'', etc., will generally *not* coincide with one another, *e.g.* that events obtaining identical t-labels may receive different t'-labels, and so on. Such, in fact, will be the case; the labels of different sorts, dashed and non-dashed, though none is privileged in any way, will have to be carefully distinguished from one another. In a word, it will appear that, with the above definition of simultaneity, no universal, no unique time-labelling is possible.

It will be enough to consider explicitly, besides S, one other system only, say, S'. Supposing that a consistent time-labelling of events occurring at different places of S' or an S'-time, is possible, like the above S-time, the question is, how is this time t' to be connected with the time t? We shall see that the connexion sought for will involve also the coordinates defining the position of points

* Which Einstein himself, in order to have a convenient name, provisionally, calls 'the stationary system.'

within S, and within S'. In a word, the time-connexion of both systems will turn out to be entangled with their space relations.

Here we shall have to appeal to what Einstein calls the principles of 'relativity' and of 'constancy of light-velocity,' and which he enunciates in the following way:

I. **The Principle of Relativity.** *The laws of physical phenomena* are the same, whether these phenomena are referred to the system S or to any other system (of coordinates) S′ moving uniformly with respect to it.*

II. **The Principle of Constant Light-Velocity.** *Every light-disturbance is propagated, in vacuo, relatively to the system S with a determinate velocity c, no matter whether it is emitted from a source (body) stationary in or moving with respect to S.* The 'velocity' c is the light-path divided by the corresponding time-interval, $c=\overline{ab}/t$; t being the S-time as defined above.

Here, to begin with and to fix the ideas, the system S is taken. But applying the principle I., we can say at once that the same constancy of light propagation is valid also with respect to the system S'. The constancy of the velocity of light, *i.e.* its independence of the motion of the source, as emphasised in Chap. II., has already been appealed to by Fresnel. But there is this essential difference that Fresnel claimed this property of light propagation only for a certain, unique system of reference, namely the aether or a system fixed in the aether, while Einstein, by accepting I. and II., postulates it for any one out of an infinity (∞^3) of systems moving uniformly with respect to one another. With regard to this property the systems S', S'', etc., are perfectly equivalent to the system S or become so in force of Principle I.,—and this is the reason why the mere notion of an 'aether' breaks down. None of the systems in question is privileged. To make it as plain as possible, let P be a point fixed in the system S, and let a point-source, moving relatively to S in a quite arbitrary manner, emit an instantaneous flash just when it is passing through P. Then the observers rigidly attached to the system S will find that the disturbance is propagated from P in all directions with the same velocity c, *i.e.* that the ensuing thin

* Literally: 'The laws according to which the states of physical systems are changing,' etc. (Einstein, p. 895).

pulse or wave of discontinuity is a spherical surface, of centre P and of radius

$$r = ct,$$

if t is reckoned from the instant of emission. Again, if P' is a point fixed in S', and if the arbitrarily moving source emits a flash just when it is passing through P', then the wave, as it appears to observers rigidly attached to S', will be a sphere whose centre is permanently situated at P' and whose radius at any instant of the S'-time is

$$r' = ct',$$

if t' is reckoned from the instant of emission. Such is, in virtue of I., the meaning of the principle of 'constancy' of light-propagation in empty space. Of especial interest is the particular case, in which our source is fixed at a point P' of the system S', and therefore moving uniformly with respect to S. In this case the centre of the spherical wave will, to the S'-observers, be permanently situated at the material particle playing the part of source, whereas for the S-observers the centre of the spherical wave, fixed in S, will detach itself from the material source, the source moving away from it with uniform velocity together with the whole system S'. This case will be made use of presently.

Let us now return to the first of the above principles, and let us remember how the time t, valid for the whole system S, has been defined. Since S has been endowed with physical properties required for a consistent method of time-labelling of events occurring at its various points, the same properties will, in virtue of I., hold also for S'. Again, local clocks satisfying the requirements of convenience, *e.g.* the causality-maxim, being possible in S, such time-keepers are, by I., possible also for various stations taken in S'. We can therefore consider first a time t_a', measured by a clock placed at a point a' in S', then distant clocks placed at b', etc., leaving the task of testing their synchronism to observers attached to the system S', and repeating in fact literally all that has been said before with regard to the system S. In this way we should obtain out of the originally local times a unique time t' applicable to the whole system S'. Let us call the time thus constructed **the S′-time.**

The question now is, how are the S'-time and the S-time connected with one another (and, possibly, with other things, viz. lengths or distances as measured by the S- and S'-observers)?

The answer to this fundamental question may be obtained, with the help of the two above principles, in a variety of ways. But for certain reasons the following way, though not the shortest, seems to me the most instructive to begin with.* It is, moreover, intimately connected with what has been said in the last chapter with regard to the Michelson experiment.

Let us imagine an S'-observer having at his disposal a point-source of light at a place P' fixed in the system S'. Let A' and B' be a pair of distant points also fixed in S', and such that the straight line $P'A'$ is in the direction of motion of S' relatively to S, and that $P'B'$ is perpendicular to that direction (Fig. 11). As before, we shall call $P'A'$ longitudinal, and $P'B'$ transversal. Let l' be the

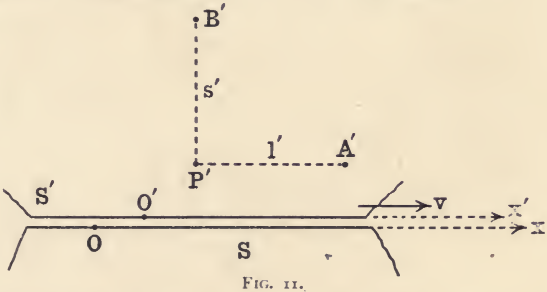

FIG. 11.

'length' of the first of these segments or the 'distance' from P' to A', according to the estimation of the S'-inhabitants, and similarly s' the length of the second segment. Suppose that our observer sends an instantaneous light-flash from P' towards A' and receives it back at P' after the lapse θ' of the t'-time. Then, having assured himself by any means that an assistant stationed at A' sends him back his signals without any delay, our observer will write

$$\theta' = \frac{2l'}{c}.$$

Under similar conditions, if he sends a flash towards† B' and

* Einstein's method of reasoning, as given in his original paper (§ 3, see also Notes at the end of this Chap.) may be mathematically interesting, but does not seem to be the fittest when a clear discussion of the physical aspect of the question is aimed at.

† To avoid unnecessary difficulties as to hitting the receiving station, now B' and now A', it will be best to imagine that our observer sends each time a full spherical wave of discontinuity or a very thin spherical pulse. This will be found especially convenient when we come next to consider the same processes from the S-standpoint.

receives it back after the interval τ' of the t'-time, he will put down the equation

$$\tau' = \frac{2s'}{c}.$$

There is, in fact, by the above principles, no difference between longitudinal and transversal light signalling between stations fixed in S', as observed by the inhabitants of this same system.

Let us now see how each of the above two processes will be described by an observer attached to the system S. Call the lengths or distances $P'A'$, $P'B'$, as estimated by the S-observer, l and s respectively. Each of these is obtained by ascertaining, with the help of an appropriate number of synchronous t-clocks, which are the points of the S-system, through which P' and A', or P' and B' pass simultaneously, and by measuring the mutual distances of these points by means of an S-standard rod. Similarly, l' and s' are to be considered as the distances $P'A'$ and $P'B'$ measured by standard rods which the S'-observers are carrying with themselves. Notice that, by Principle I., l' and s', thus measured, will be the same whether the system S', together with its observers, clocks and measuring rods, is at rest with respect to S or whether it moves uniformly with respect to that system, as it actually does. But l, s are not necessarily equal to l', s'. For although they are 'distances of the same pairs of material points,' the source and the receiving stations, they are not obtained by the same processes. Having thus explained the meaning of l, s, let us consider, from the S-standpoint, first the longitudinal and then the transversal signalling. The flash sent out by the luminous source will, according to Principle II., appear to the S-observers in both cases as a spherical wave expanding with the velocity c and having its centre at that point P_0, fixed in S, through which the source has passed when emitting the flash. Now, if v be the velocity of S' relative to S, the receiving station A' moves away from P_0 with the uniform velocity v. If, therefore, θ_1 be the S-time required for the wave to expand from P_0 to A',

$$c\theta_1 = l + v\theta_1,$$

and

$$\theta_1 = \frac{l}{c-v}.$$

In the same way, if θ_2 be the S-time employed by the light

to return from the receiving station* to the sending station P',

$$\theta_2 = \frac{l}{c+v}.$$

Thus, the S-time θ elapsing between the first appearance and the reappearance of a light flash at A', being the sum of θ_1 and θ_2, will be given by

$$\theta = \frac{2lc}{c^2 - v^2}. \quad = 2\gamma^2 \frac{l}{c}$$

A similar reasoning applied to the case of transversal signalling, in which case the sphericity of the wave will be found particularly convenient, will give us for the S-time elapsing between the appearance of the first and second flash at A' the value

$b = d\frac{v}{c} = d\beta$

$s^2 = d^2 - d^2\beta^2 = d^2(1-\beta^2)$

$d = \frac{s}{\sqrt{1-\beta^2}} = s\gamma$

$$\tau = 2\gamma \frac{s}{c}, \qquad 2d = \tau c, \quad \tau = \frac{2d}{c} = 2\gamma\frac{s}{c}$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, $\beta = v/c$. $\gamma > 1$, $\beta < 1$

Compare the last two formulae with the above ones for θ' and τ', and denote the ratio s/s' by α. Then the result will be

$\gamma^2 = \frac{c^2}{c^2 - v^2}$

$$\frac{\theta}{\theta'} = \gamma^2 \frac{l}{l'}; \quad \frac{\tau}{\tau'} = \gamma\alpha; \quad \frac{s}{s'} = \alpha, \tag{3}$$

for $a = 1$, $\tau = \gamma\tau'$
therefore for an S observer even the transverse

where α is a number which for $v = 0$ becomes equal 1, but is otherwise an unknown function of the data of the problem.

Now, each of the two processes, *i.e.* the longitudinal and the transversal signalling, may (by disregarding the receiving stations) be considered as phenomena consisting in a double appearance of a flash *at one and the same station*, at the same individually discernible point A', fixed in S'. Thus far we have, purposely, kept these two processes separate. But now we can advantageously combine them with one another. If the receiving stations were chosen so that $s' = l'$, then we should have, by the first pair of formulae,

$$\theta' = \tau', \quad \text{say} \quad = T',$$

and if the two processes were started simultaneously, from the S'-point of view, they would also have ended simultaneously for the S'-inhabitants. In other words, we would have, in S', a pair of

* This station A' (and similarly, in the case of transversal signalling, the station B') may be imagined to become an instantaneous point-source emitting a spherical wave at the moment when it is reached by the original wave.

simultaneous events followed by another pair of simultaneous events, all of these occurring at the same place A'. Let us now require (what, as far as I know, is tacitly assumed by most authors) that

III. *Events* **locally*** *simultaneous for an S'-observer should also be simultaneous for the S-observers.* this forces the inequalities upon the lengths and times, see below

This amounts to supposing that there is a *one-to-one correspondence* between the t-labels and the t'-labels to be applied to events occurring at any given place, *i.e.* for *fixed* values of the coordinates x', y', z' in S'. (The analogous one-to-one correspondence between x', y', z' and x, y, z for $t'=$const. is tacitly assumed as a matter of course.) On the other hand, two events occurring at distinct places, being simultaneous in S', are generally *non*-simultaneous from the S-standpoint.

Now, in virtue of the requirement III., call it postulate or desideratum, or whatever you prefer, the above two simultaneous processes or phenomena occurring at A' will also begin and end simultaneously for the S-observers, so that

$$\theta=\tau, \quad \text{say} \quad =T,$$

and

$$\theta/\theta'=\tau/\tau'=T/T'.$$

Consequently, by the equations (3),

$$\left.\begin{aligned} T/T'&=\alpha\gamma=\alpha(1-\beta^2)^{-\frac{1}{2}} \\ l/l'&=\alpha\gamma^{-1}; \quad s/s'=\alpha. \end{aligned}\right\} \tag{4}$$

These are the required connexions between durations and lengths, measured in S and in S'. They are based on the above assumptions I., II., III., the last of which is certainly the most obvious. The common coefficient α is, thus far, indeterminate. If we are to endow (empty) space with homogeneity, as well as with isotropy,† and if it be granted that the relations between the S- and S'-measurements do not vary in time, the unknown coefficient α can depend only upon $v=c\beta$. The only thing we thus far know about this

* *i.e.* occurring at one and the same place.

† Both properties having been already attributed to it physically, *i.e.* as regards propagation of light, by II.

function is that it reduces to unity for $\beta = 0$, when S' is at rest relatively to S, when, in fact, both systems cease to be discernible from one another. Thus

$$\alpha = \alpha(\beta), \quad \alpha(0) = 1.$$

Notice that for $v = 0$ we have also $\gamma = 1$, so that in this case T, l, s become, by (4), identical with T', l', s', as was to be expected.

To put the relations (4) in words as simply as possible, and to fix the ideas, let us assume for the moment $\alpha = 1$. Then

(? therefore $\beta = 0$, $v = 0$?)
No, not necessarily.

$$\frac{T}{T'} = \gamma; \quad \frac{l}{l'} = \frac{1}{\gamma}; \quad s = s'. \tag{4a}$$

for $a = 1$

Thus, a transversal bar s' sharing the motion of S' will have the same length from the standpoint of either of the two systems S, S', while a bar of longitudinal orientation and of length l' in S' will, according to the estimation of the S-observers (with equal t-values for both terminals of the bar), be shortened to $l = l'\sqrt{1 - \beta^2}$. A solid fixed in S', which for the inhabitants of that system is a sphere of radius R, will, according to the estimation of the S-observers, become a longitudinally flattened ellipsoid of semi-axes

$$\frac{1}{\gamma}R, \; R, \; R,$$

precisely as in the *contraction hypothesis* of Fitzgerald and Lorentz. It is a slightly different thing to say, instead of this, that a body which for the S-observers is spherical while at rest in S becomes flattened down to the above ellipsoid when set in motion with the translation-velocity v relative to S. The clause hinted at is in connexion with the manner in which the body is set from rest to motion and cannot satisfactorily be dealt with at this stage of our considerations. Again, as regards the ratio of times, remember that T' is the S'-duration of a phenomenon or process going on at a place P' fixed in S', *i.e.* for constant x', y', z'. This duration or time-interval is then lengthened in the estimation of the S-observers to $T = \gamma T' = T'/\sqrt{1 - \beta^2}$. We are assuming here, of course, that $\beta < 1$, so that γ is real and greater than unity. Instead of a pair of flashes, as considered above, we may think of two consecutive indications of an S' clock preserved at P', and we may say that a clock moving relatively to S with the uniform velocity v *goes slower*, in the

ratio $\sqrt{1-\beta^2} : 1$, than 'the same' clock when at rest in S. This at least is the way that the leading relativists put the above result. 'The same' is taken to mean that the mechanism of the clock has undergone no changes due to its passing from rest to motion, except those which are implied by the fundamental relativistic principles themselves. This statement does by no means look satisfactory, but it can be made more rigorous and clear by returning to it after certain portions of relativistic physics have been worked out. The practically important question is, which are the physical systems we are going to consider as such clocks whose 'internal mechanism' is not subject to changes due to their merely passing from rest to uniform motion relatively, say, to the earth or the fixed stars. Now, as far as I know, the prevailing tendency is to consider as such physical systems the various atoms (or at least, if they are to serve us for thousands of years, those which are not sensibly radioactive) with their 'natural' periods of vibration, manifested in their characteristic spectrum lines.* The influence felt by such minute mechanisms in the presence of a strong magnetic field (Zeeman's effect) will not, of course, be forgotten. Who knows but that some remote future generations, to get rid of such physical influences, may choose to consider as 'invariable' the mechanism not of light emission but of radioactive disintegration of atoms. If such is to be the case, the formula $T=\gamma T'$ will be interpreted by saying that the 'half-life' of radium, which is about 1760 years, is in the estimation of a terrestrial observer lengthened by a month or so, when flashing before him with something like one hundredth of the velocity of light.

We have already remarked in passing that two events occurring simultaneously in S' at places *distant* from one another will generally be non-simultaneous to the S-observers. This may be seen immediately by the principle of constant light-velocity, valid by I. for both S and S'. For let a spherical wave or a very thin pulse be started from our point-source placed at P'. Then, if $l'=s'$, the arrivals of flashes at A' and B' will be a pair of events simultaneous

* Thus we read, in M. Laue's *Relativitätsprinzip*, second edition, 1913, p. 42: 'In einem bewegten Wasserstoffatom (Kanalstrahlen) werden, zum Beispiel, die Licht emittierenden Eigenschwingungen geringere Frequenz haben, als in einem ruhenden.'

As regards the experimental side of the subject, see J. Laub's report in *Jahrb. d. Rad. u. Elektronik*, Vol. VII. p. 439.

to the S'-observers. On the other hand, the S-time required for the wave to reach A' will be

$$T_{P'A'} = \frac{l}{c-v},$$

and that to reach B'

$$T_{P'B'} = \gamma \frac{s}{c}.$$

Now, by (4*a*), and also by the more general formulae (4),

$$l/s = \gamma^{-1} l'/s' = 1/\gamma,$$

whence

$$T_{P'A'} - T_{P'B'} = \frac{\beta\gamma}{c} s. \quad \left(= \frac{l}{c-v} - \gamma \frac{s}{c};\right.$$

substitute for v: $c\beta$
for $l = \frac{s}{\gamma}$
remember $\frac{1}{\gamma^2} = 1 -$

Thus, the pair of events in question will not be simultaneous for the S-observers. Instead of the two particular points A', B', the whole wave may be considered. Then it will be seen at once that the sphere $r' = \text{const.}$ with centre P' will, to the S'-observers, be the *locus* of points reached simultaneously by the wave, but not so to the S-observers. For to these the *loci* of simultaneously illuminated points will be spheres centered at a point, P_0, fixed in S, from which P' is continuously moving away.

Thus, the notion of distant simultaneity, to call it again by this short name, has no 'absolute' or universal meaning, but involves a specification of one out of ∞^3 systems of reference. For such is the manifold of the vector-values of their relative velocity $\mathbf{v}$, its absolute value v amounting to one scalar, and its direction to two more.

TRANSFORMATION

Let us now once more return to our formulae (4), with the view of deducing from them the relations connecting the S-time and coordinates t, x, y, z with the S'-time and coordinates t', x', y', z'. Take the x'-axis coincident in direction and sense with the x-axis, both concurrent with the vector $\mathbf{v}$ fixing the velocity of S' relative to S (Fig. 11),* and the axes of y', z', both transversal and perpendicular to one another, parallel to and concurrent with the axes of y, z respectively. Count both the S'- and the S-time from the

* In that figure the systems S', S are represented as sliding along one another only to avoid confusion in the drawing, but in reality they are to be imagined as interpenetrating one another throughout the whole (three-dimensional) space.

instant at which the origins of the coordinates, O' and O, coincide with one another, *i.e.* assume

$$t = x = y = z = 0$$

as corresponding to

$$t' = x' = y' = z' = 0,$$

which is a pure convention. The axes of y' and z' will then coincide at that instant with the axes of y and z. Let us fix our attention on any point $P'(x', y', z')$ taken in S'. Then by the second of formulae (4), in which we have to write $l = x - vt$, $l' = x'$,

$$x' = \frac{\gamma}{\alpha}(x - vt), \tag{5}$$

and, by the third of those formulae,

$$y' = \frac{1}{\alpha} y; \quad z' = \frac{1}{\alpha} z. \tag{6}$$

To obtain t as a function of x', y', z', t', notice first of all that events occurring at various points of a *transversal* plane ($x' = \text{const.}$), being simultaneous in S', are also simultaneous with one another according to the S-point of view. For if M', N' be a pair of such points, and if $\overline{M'N'} = s'$, then a wave started at their mid-point C' at the instant $t' - \frac{1}{2}s'/c$ will reach both M' and N' simultaneously, at the instant t'. Again, from the S-standpoint, in our previous notation,

see p 103, $\tau = 2\gamma\frac{s}{c}$ for transverse signal

from S-standpoint

$$T_{C'M'} = \frac{\gamma s}{2c} = T_{C'N'},$$

from S′-standpoint. $T_{C'M'} = \frac{s}{2c} = T_{C'N'}$

times of arrival

so that M' and N' will receive the signals at the same instant t. Thus, t is independent of y', z', and consequently

$$t = t(x', t').$$

Next, take a longitudinal pair of points, say P' on the x'-axis and the origin O'. Call x' the abscissa of P'. Imagine a wave started at the mid-point of O' and P' at the instant $t' - \frac{1}{2}x'/c$; then the wave will reach O' and P' at the same instant t', and, by Principle II. and by the second of formulae (4),

$$t(x', t') - t(0, t') = \frac{\alpha}{\gamma} \frac{x'}{2} \left(\frac{1}{c - v} - \frac{1}{c + v} \right) = \alpha\gamma \frac{v}{c^2} x'.$$

But, by the first of formulae (4) and by the above convention as to the origin of time-reckoning at O',

$$t(0,\ t') = \alpha\gamma t'.$$

Hence

$$t \equiv t(x',\ t') = \alpha\gamma\left(t' + \frac{v}{c^2}x'\right), \tag{7}$$

which is the required connexion. Substituting here x' from (5) and remembering that $\beta^2 + 1/\gamma^2 = 1$, we shall obtain t' in terms of t, x. also $1+\gamma^2\beta^2=\gamma^2$ and that $\frac{v^2}{c^2}=\beta^2$, or $1+\gamma^2\frac{v^2}{c^2}=\gamma^2$

Thus, the complete set of formulae connecting the S'- with the S-time and coordinates will be

$$\left.\begin{array}{c} x' = \dfrac{\gamma}{\alpha}(x - vt)\,; \quad y' = \dfrac{1}{\alpha}y\,; \quad z' = \dfrac{1}{\alpha}z \\ t' = \dfrac{\gamma}{\alpha}\left(t - \dfrac{v}{c^2}x\right). \end{array}\right\} \tag{8}$$

Conversely, resolving these equations with respect to t, x, y, z, or simply copying (7) and using it to eliminate t from the first equation,

$$\left.\begin{array}{c} x = \alpha\gamma(x' + vt')\,; \quad y = \alpha y'\,; \quad z = \alpha z' \\ t = \alpha\gamma\left(t' + \dfrac{v}{c^2}x'\right). \end{array}\right\} \tag{9}$$

Notice that, disregarding α, the set (9) follows from (8), and *vice versa*, by simply interchanging x, y, z, t with x', y', z', t' and by writing $-v$ instead of v. Now $\mathbf{v}$ being the velocity of S' relative to S, $-\mathbf{v}$ will be the velocity of S relative to S'.* As to c, it is common to both systems, and $\gamma(v) = \gamma(-v) = (1 - v^2/c^2)^{-\frac{1}{2}}$. Thus, there is **reciprocity** between the two systems of reference, except for the common arbitrary coefficient which is α^{-1} in the

* In fact, what we call the velocity of S relative to S' is the vector whose components are the derivatives of x', y', z' with respect to t', *for constant* x, y, z, that is to say, by (8),

$$\frac{dx'}{dt'} = -v, \quad \frac{dy'}{dt'} = 0, \quad \frac{dz'}{dt'} = 0,$$

and this is the vector $-\mathbf{v}$. In exactly the same way, the velocity of S' relative to S is the vector whose components are the derivatives of x, y, z with respect to t, *for constant* x', y', z', *i.e.*, again by (8),

$$\frac{dx}{dt} = v, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 0.$$

first, and a in the second set of formulae. As a matter of fact, there is a *physical* reciprocity anyhow, *i.e.* for any $a=a(v)$, subjected to the condition $a(0)=1$. For the conditions imposed upon the time-labellings in S and in S', in order to make them self-consistent, will continue to be satisfied when all values of time and coordinates, in S or in S', have been multiplied by a common factor; a^{-1} in one, and a in the other case may be thrown back upon the choice of the units of measurement. Thus, the choice of a being a matter of indifference, we *may* take $a=1$. But, if not content with the physical, we require also a formal reciprocity, then we *have* to write

$$a^{-1}=a, \quad i.e. \quad a^2=1.$$

But $a(0)=1$. Thus, if $a(v)$ is to be continuous, $a=+1$.*

In this way we obtain the formulae of what is universally called the **Lorentz transformation**,

$$\left.\begin{aligned} x'&=\gamma(x-vt); \quad y'=y; \quad z'=z \\ t'&=\gamma\left(t-\frac{v}{c^2}x\right), \end{aligned}\right\} \qquad (10)$$

already met with in Chap. III. But here, as can be judged from the whole line of reasoning, the meaning and the rôle of this transformation are essentially different from what they were in Lorentz's theory, based as it was on the assumption of a privileged system of reference, the aether.

Let us write also the inverse transformation

$$\left.\begin{aligned} x&=\gamma(x'+vt'); \quad y=y'; \quad z=z' \\ t&=\gamma\left(t'+\frac{v}{c^2}x'\right). \end{aligned}\right\} \qquad (10')$$

The above postulate I., or the Principle of Relativity, may now be expressed in the concise and more definite form:

I^a. *The laws of physical phenomena, or rather their mathematical expressions, are* **invariant** *with respect to the Lorentz transformation.*†

* With regard to Einstein's own treatment of this subject, and also that adopted in Laue's book, see **Note 1** at the end of the present chapter.

† Some authors employ in this connexion the mathematically sanctioned term *covariant*, instead of invariant. But it will be convenient to reserve 'covariant' for another use, namely to denote that two groups of magnitudes are equally transformed.

That is to say, if a law L, valid in S, involves—besides other magnitudes—x, y, z, t in a certain way, and if these are transformed according to (10), then the resulting law L', valid in S', will involve x', y', z', t' in exactly the same way. Any system S', with its corresponding tetrad of independent variables, is as 'legitimate' as S. The choice of one out of ∞^3 systems of reference moving uniformly with respect to one another is a matter of indifference. As regards the behaviour of the 'other magnitudes' involved in the laws, any attempt to elucidate it by general remarks in this place would be useless. We shall come to understand this point by and by when considering various applications of the above principle. And, with regard to the specification 'physical,' it has, of course, to be taken in the broadest sense of the word. The phenomena in question may as well be chemical or physiological (though, for the present, physiology is far from being prepared to receive a theory of such a high degree of accuracy). Instead of 'physical phenomena' the reader can, at any rate, put theoretically: any phenomena which are at all localizable in space and in time. But subtleties of this kind need not detain us here any further.

The principle of relativity excludes all such laws as are not invariant with respect to the Lorentz transformation. Thus, for instance, Newton's inverse square law of universal gravitation, or even his general laws of motion, cannot stand in their original form, but require some slight modifications, if they are to be brought into line with the principle in question. But there is certainly no need to multiply such negative examples; the reader can pick out at random as many cases as he wants, and he is sure never to hit a case which does not contradict the principle of relativity. Maxwell's equations for the 'free aether,' also with the supplementary term $\rho\mathbf{p}$, and for 'stationary' ponderable media, are, as has been already remarked, in an exceptional position. But these electromagnetic equations will occupy our special attention in later chapters.

Thus far we have had only one example of a 'law' which is proclaimed to be *rigorously valid*, with reference to S, namely the law of light propagation, as enunciated in the principle of constant light-velocity.* Thus, the true office of II. is to fix a particular case of a physical law which is postulated rigorously to satisfy I.

*Notice that, in considering this law, we need not yet trouble about the electromagnetic, or any other, theory of light.

This law then has certainly to be invariant with respect to the Lorentz transformation. And since this transformation has been obtained by means of the law itself, applied both to S and S', it can be foreseen without calculation that this law will prove to be invariant. In fact, this prevision may be verified at once. For the law in question states that if light be emitted at the instant $t=0$ by a point-source, placed at or just passing through a given point, which may be taken as the origin of the coordinates, O, then at any instant $t>0$ it reaches a spherical surface of radius $r=ct$ and centre O, *i.e.* such that, x, y, z being the coordinates of that surface,

$$x^2+y^2+z^2-c^2t^2=0. \tag{11}$$

Now, squaring the equations (10) and adding up, we have, identically,

$$x'^2+y'^2+z'^2-c^2t'^2=x^2+y^2+z^2-c^2t^2, \tag{12}$$

and consequently also

$$x'^2+y'^2+z'^2-c^2t'^2=0. \tag{11'}$$

Thus, the law of light propagation, (11), is invariant with respect to the Lorentz transformation. Remember that O' coincides with O for $t=0$, when also $t'=0$, and that, therefore, (11′) expresses for S' precisely the same thing as (11) for S. Notice, moreover, that the law under consideration would be invariant with any value of α (not zero). For, then, we should have, by (8),

$$x^2+y^2+z^2-c^2t^2=\alpha^2(x'^2+y'^2+z'^2-c^2t'^2),$$

and what we require is not so much the invariance of the quadratic function $x^2+y^2+z^2-c^2t^2$ as that of the equation (11). But having once decided, be it only for purely formal reasons, to take $\alpha=1$, the property (12), which will in the sequel be often referred to, is worth keeping in memory.

It may be expressed shortly by saying that

$$x^2+y^2+z^2-c^2t^2, \quad \text{or} \quad r^2-c^2t^2$$

is a **relativistic invariant.** Any function of this expression alone is, of course, again an invariant. But all of these count as one invariant. It is worth noticing that, on this understanding, there are among all functions of x, y, z, t no other invariants than

$$x^2+y^2+z^2-c^2t^2.$$

In what precedes we have used the integral form, (11), of (a particular case of) the law of propagation. We might as well have used its differential form,

$$\Box\phi = 0, \tag{13}$$

where ϕ may be thought of as any one of the rectangular components of a 'light-vector,' and where

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{14}$$

is Cauchy's symbol, called also the **Dalembertian.** The physical meaning of this famous differential equation is (among other things) that any element of a wave of discontinuity is propagated normally to itself with the velocity c (cf. **Note 2**). This then is the general law of which the previous is but a particular case, corresponding to a particular form of the wave. Now, by (10),

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \beta\frac{\partial}{c\,\partial t'}\right); \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'};$$

$$\frac{\partial}{c\,\partial t} = \gamma\left(\frac{\partial}{c\,dt'} - \beta\frac{\partial}{\partial x'}\right),$$

so that

$$\Box = \Box', \tag{15}$$

which proves the invariance of the differential law of the propagation of light in empty space. But since (13) involves further particulars not yet entered upon (embodied summarily in ϕ) concerning light, the reader is recommended to keep rather to the above integral form (11), until we come to consider the relativistic properties of electromagnetic laws. Meanwhile he is asked to retain in memory solely that *the Dalembertian is an invariant* as good as $r^2 - c^2t^2$, although the latter is a magnitude and the former an operator.

Conversely, the Lorentz transformation may be obtained by postulating the invariance of the Dalembertian and by making some auxiliary assumptions (**Note 3**). But the above method of obtaining the transformation formulae seemed to me to be more suitable for bringing into prominence their physical meaning.

Basing ourselves upon the Principles I., II., and upon the obvious requirement III., we have obtained the formulae (4a) for the ratios of time-intervals and lengths as measured in S and S'. From these formulae the Lorentz transformation (10), and its inverse

(10′), followed almost immediately. Now, it may be well to notice here how (4*a*) are to be obtained conversely from (10), (10′). The third of (4*a*) is identical with $y=y'$, $z=z'$. To obtain the first of formulae (4*a*), remember that it was valid for a point (any point) fixed in S'. Take therefore, in the last of (10′), $x'=$const., and denote by Δ any increment. Then the result will be

$$\Delta t=\gamma\,\Delta t'.$$

Similarly, remembering that the terminals of the segment l are to be taken simultaneous in S, take, in the first of (10), $t=$const.; then the result will be

$$\Delta x=\frac{1}{\gamma}\Delta x'.$$

Now, these are precisely the relations stated by (4*a*). Notice that the constancy or variability of the transversal coordinates y, z is a matter of indifference. As to the fact, mentioned on several occasions, that simultaneous events occurring at distant places in S' are generally not simultaneous in S, and *vice versa*, it is most immediately expressed in (10), (10′) by the circumstance that t contains x' besides t', and similarly, that t' contains x besides t.

So long as $v<c$, or $\beta<1$, the coefficient γ is real and greater than unity, so that the duration of any process, local in S', is lengthened, to the S-observers, γ times or in the ratio $1:(1-\beta^2)^{\frac{1}{2}}$, and any longitudinal segment $\Delta x'$ is contracted to $\Delta x'(1-\beta^2)^{\frac{1}{2}}$. In the critical case of $v=c$, or $\beta=1$, we have $\gamma=\infty$. Then any finite duration $\Delta t'$ becomes infinite in S, and any finite distance $\Delta x'$, as judged by the S-observers, dwindles down to nothing: the whole of S', with all the bodies sharing its motion, becomes a transversal flatland. Finally, for $\beta>1$, *i.e.* when the velocity of S' relative to S exceeds the velocity of light or when it becomes what may conveniently be called a **hypervelocity**,* γ is purely imaginary and so also are x, t for any real values of x', t'. But, as far as I can see, this does not necessarily mean that motion with hypervelocity, of one body relative to another, is 'impossible.' It would, thus far, be enough to say simply that there is in this case no correlation in real terms between S' and S to be obtained by light-signalling. Notice that, from the S-standpoint, any station P' can then succeed in sending light-signals only to points contained in a certain back-

*The Germans call it 'Ueberlichtgeschwindigkeit.'

cone, so that, according to that standpoint, no such station can ever receive back any of its signals, and that therefore the whole of our previous reasoning ceases to be applicable to the case in question. In what sense hypervelocities are, or by what reasons they may be required to be, 'impossible,' will be seen from the physical applications of the principle of relativity.

For the present, and for what follows, we shall simply assume

$$v < c,$$

considering only now and then the limiting value $v=c$.

To touch the other extreme, let us suppose that v is a very small fraction of c. Then, neglecting β^2-terms, and limiting ourselves to such values of x as are not enormously great compared with ct, we obtain from (10) the Newtonian transformation (Chap. I.)

$$x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t.$$

Or, if we like, we can say also that, if ∞ is taken instead of c, the Lorentz transformation reduces to the Newtonian transformation. Just as the equations of classical or Newtonian mechanics were invariant with respect to the Newtonian transformation, so are the fundamental laws of optics and (as we shall see later) of electromagnetism invariant with respect to the Lorentz transformation. Let us call the principle associated with the former *the classical principle of relativity*, and that corresponding to the latter of these transformations *the new principle of relativity*. Then it is obvious that we cannot have both, retaining the classical principle for our mechanics and using the new one for our electromagnetism. For if S be a particular system or space-framework of reference in which the laws of both classical mechanics and electromagnetism are valid, then, among all the systems moving with respect to it with uniform velocity, no other would have this property.* In other words, the system S would be privileged, being *the* system for both classes of laws, whereas, according to the general principle of relativity, *i.e.* according to I. taken by itself (without yet touching II.), none of the manifold of ∞^3 systems moving uniformly with respect to one another is to be privileged, equal rights being claimed for all of them with regard to any physical phenomena. Thus, if we are

*Supposing, of course, that the inhabitants of each system avail themselves only of *one* set of coordinates and time.

to construct a truly relativistic theory at all, we can have *but one Principle of Relativity*, that is to say, one at a time. (It may well happen that the next, or even the present, generation will have to give up the 'new' principle for a yet broader one.) Now, Hertz's and Heaviside's attempts to extend the classical principle of relativity to the domain of electromagnetism proved a complete failure. And since, for the time being, *tertium non datur*, the 'new' principle, involving the Lorentz transformation, has become *the* principle of relativity of modern physics. In this connexion it must not be forgotten that electromagnetic and especially optical phenomena have been known all these years with a much higher degree of accuracy than the various instances of motion of material bodies. No wonder, therefore, that the physicist has so easily decided to mould his mechanics and thermodynamics according to a principle which sprang out from optical and, generally, electromagnetic ground. This is not to say, of course, that mechanical and all other phenomena must be 'ultimately' electromagnetic, *i.e.* that everything must be explained by, or reduced to, electromagnetism. The theory of relativity is not concerned at all with such reduction of one class of phenomena to another. It does not force upon us an electromagnetic view of the world any more than a mechanical view. Quite the contrary; it opens before us a wide field of possibilities of asserting that even the mass of a free electron, say a β-particle, must not be entirely electromagnetic.

Like the Newtonian transformations, the Lorentz transformations, generally with the inclusion of pure space-rotations,* constitute a **group**, that is to say, two of such transformations applied successively one after the other are equivalent to a single transformation, which is again a Lorentz transformation. In the case of the Newtonian transformation, if $\mathbf{v}_1$ be the velocity of S' relative to S, and $\mathbf{v}_2$ the velocity of S'' relative to S', the vectorial parameter $\mathbf{v}$ of the resultant transformation is simply the sum of the parameters of the component transformations, *i.e.* $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. The parameter of the resultant Lorentz group is a more complicated function of the parameters of the component transformations, thus leading to a more complicated rule of physical addition of velocities, which will be given in the next two chapters. Only when the absolute values of $\mathbf{v}_1$, $\mathbf{v}_2$ are small compared with the critical velocity, does the familiar rule of composition of velocities reappear. Classical kine-

* This reservation will become clear in Chapter VI.

matic is but a limiting case of modern relativistic kinematic. So are also most of the remaining branches of mechanics and generally of physics. For slow motion the reader will recognise throughout his good old friends in this new and strange land of relativistic connexions.

To close this somewhat lengthy chapter on the foundations of the theory of relativity, one short remark more. Einstein's results concerning electromagnetic and optical phenomena will be seen to agree in the main with those which have been obtained by Lorentz in his generalized theory, the chief difference being (to quote Lorentz's own words, *Columbia University Lectures*, p. 230) that Einstein simply *postulates* what Lorentz has deduced 'with some difficulty, and not altogether satisfactorily, from the fundamental equations of the electromagnetic field. By doing so, he may certainly take credit for making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but the manifestation of a general and fundamental principle. . . . It would be unjust not to add that, besides the fascinating boldness of its starting point, Einstein has another marked advantage over mine. Whereas I have not been able to obtain for the equations referred to moving axes *exactly* the same form as for those which apply to a stationary system, Einstein has accomplished this by means of a system of new variables slightly different from those which I have introduced.' (As to these slight differences, cf. Note 86 to Lorentz's *Lectures*.)

We see from the above quotation that Lorentz himself aimed at an exact sameness of form of the laws of all, or at least of electromagnetic, phenomena for a pair of systems moving uniformly with respect to one another. Why then not postulate this sameness at once? But Lorentz had not the heart to abandon the aether which he confessedly 'cannot but regard as endowed with a certain degree of substantiality.'

NOTES TO CHAPTER IV.

Note 1 (to page 110). Einstein, *Ann. d. Physik*, Vol. XVII., 1905, § 3, obtains the formulae of transformation (10) in the following way:

Let, to use our notation, x, y, z, t be the coordinates and the time in S, and x', y', z', t' those in S'. Write

$$\xi = x - vt\,;$$

then to a point fixed in S' corresponds a system of values of ξ, y, z independent of t. To obtain t' as a function of ξ, y, z, Einstein considers a signal sent at the instant t_0' from the origin O' along the axis of x' towards the point ξ, where it arrives at the instant t_1, and, being reflected there, returns to O' at the instant t_2'. Then, according to the definition of synchronism, (1*a*), p. 95, which is to hold equally for S' as for S,

$$t_1' = \tfrac{1}{2}(t_0' + t_2'),$$

i.e. filling in the arguments and applying the principle of constant light propagation,

$$t'(0, 0, 0, t) + t'\left(0, 0, 0, \left[t + \frac{\xi}{c-v} + \frac{\xi}{c+v}\right]\right) = 2t'\left(\xi, 0, 0, t + \frac{\xi}{c-v}\right),$$

whence, for an infinitesimal ξ,

$$\frac{\partial t'}{\partial \xi} + \frac{v}{c^2 - v^2}\frac{\partial t'}{\partial t} = 0.$$

Applying the same reasoning to signals sent along the axes of y or z, Einstein obtains

$$\frac{\partial t'}{\partial y} = 0, \quad \frac{\partial t'}{\partial z} = 0,$$

and, assuming t' to be a *linear* function of its arguments,

$$t' = \phi(v)\,.\left(t - \frac{v\xi}{c^2 - v^2}\right),$$

where $\phi(v)$ is thus far an unknown function of v, and where $t' = 0$ has been put at O' for $t = 0$.

Next, to obtain from the last equation x', y', z' in terms of x, y, z, t, Einstein writes the principle of constant light-propagation in S'. A signal started at O' at the instant $t' = 0$ reaches at the instant t' a point of the positive x'-axis, for which

$$x' = ct' = \phi(v)\,.\,c\left(t - \frac{v\xi}{c^2 - v^2}\right).$$

But the same process, if considered from the S-standpoint, gives $\xi = t(c-v)$. Thus

$$x' = \phi(v) \frac{c^2}{c^2 - v^2} \xi = \phi(v) \gamma^2 \xi.$$

Similarly

$$y' = ct' = \phi(v) . c \left(t - \frac{v}{c^2 - v^2} \xi \right),$$

where $t = y(c^2 - v^2)^{-\frac{1}{2}}$, $\xi = 0$. Thus

$$y' = \phi(v) \gamma y$$

and

$$z' = \phi(v) \gamma z.$$

Consequently, writing again $\xi = x - vt$, and throwing the common factor γ upon $\phi(v)$,

$$x' = \phi(v) . \gamma (x - vt), \quad y' = \phi(v) . y, \quad z' = \phi(v) . z,$$

$$t' = \phi(v) . \gamma \left(t - \frac{v}{c^2} x \right).$$

These are identical with the formulae (8) of the present chapter, for $\phi(v) = 1/a$ The way that Einstein obtains the particular value $\phi(v) = 1$ (*loc. cit.* pp. 901-902) need not detain us here. We know that the value of such a common coefficient is essentially, from the physical standpoint, a matter of indifference.

As to Laue (*Das Relativitätsprinzip*, 2nd edition, p. 38, etc.), his method of obtaining the Lorentz transformation consists in postulating the invariance of the 'wave-equation'.

$$\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = 0$$

and in assuming linearity and symmetry round the axis of motion, *i.e.* in writing

$$\left. \begin{aligned} x' = \kappa(v) . (x - vt), \quad y' = \lambda(v) . y, \quad z' = \lambda(v) . z \\ t' = \mu(v) . t - \nu(v) . x, \end{aligned} \right\} \qquad (a)$$

where κ, λ, μ, ν are functions of v alone. These functions are then easily determined from the postulated invariance which Laue writes

$$\nabla'^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} = a \left\{ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right\}, \qquad (b)$$

where a is again an unknown function of v alone. The value of λ is easily shown to be equal to unity, by requiring reciprocity, *i.e.*

$$y = \lambda(-v) . y', \quad z = \lambda(-v) . z',$$

and by remembering that 'for the y- and z-directions it is exactly the same thing whether S' moves relatively to S in the positive or in the negative sense of the x-axis,' so that $\lambda(v)=\lambda(-v)$. Thus $y'=y$, $z'=z$, and, by (b), $\kappa=\mu=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}=\gamma$, $\nu=\frac{v}{c^2}\gamma$. Substituting these values in (a), Laue obtains the required formulae (10). The discussion of Laue's method of obtaining for α the particular value 1, rather than any other, is again left to the reader.

Note 2 (to page 113). Let the function ϕ, satisfying the equation $\Box\phi=0$, be continuous, as well as its first derivatives $\partial\phi/\partial t$, $\partial\phi/\partial x$, etc., that is to say, let

$$[\nabla\phi]=0, \quad \left[\frac{\partial\phi}{\partial t}\right]=0,$$

but let the derivatives of the second order, $\partial^2\phi/\partial t^2$, $\partial^2\phi/\partial x^2$, etc., experience a discontinuity at the surface σ. Then, $\mathbf{n}=\mathbf{i}n_1+\mathbf{j}n_2+\mathbf{k}n_3$ being the normal of any surface-element $d\sigma$, at the instant t, the *identical conditions* and the kinematic conditions of *compatibility*, expressing that σ is neither split into two or more surfaces, nor dissolved, at the next instant $t+dt$, are (cf. *Ann. der Physik*, Vol. XXIX., 1909, p. 524)

$$\left[\frac{\partial^2\phi}{\partial x^2}\right]=n_1^2\lambda, \quad \left[\frac{\partial^2\phi}{\partial y^2}\right]=n_2^2\lambda, \quad \left[\frac{\partial^2\phi}{\partial z^2}\right]=n_3^2\lambda,$$

$$\left[\frac{\partial^2\phi}{\partial t^2}\right]=\mathfrak{v}^2\lambda,$$

where $\mathfrak{v}$ is the velocity of propagation, along $\mathbf{n}$, and λ a scalar characterizing the discontinuity. Now, $\mathbf{n}$ being a unit vector,

$$[\nabla^2\phi]=\lambda, \quad \text{and} \quad [\Box\phi]=\lambda\left(1-\frac{\mathfrak{v}^2}{c^2}\right)=0,$$

whence $|\mathfrak{v}|=c$. Q.E.D.

In electromagnetism ϕ has in turn the meaning of the components of the electrical and the magnetic vector, and the *sense* of propagation, $\pm\mathbf{n}$, follows from the mutual relations of these two vectors.

Note 3 (to page 113). Postulate the invariance of the Dalembertian, *i.e.*

$$\Box'=\Box,$$

and assume

$$y'=y, \quad z'=z,$$

or make any set of plausible assumptions leading to this. Then

$$\partial^2/\partial y'^2=\partial^2/\partial y^2, \quad \partial^2/\partial z'^2=\partial^2/\partial z^2,$$

and

$$\frac{\partial^2}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}=\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Instead of x, t introduce new independent variables

$$\xi = x - ct,$$

$$\eta = x + ct,$$

and similarly, for the system S',

$$\xi' = x' - ct',$$

$$\eta' = x' + ct'.$$

Then the required invariance will assume the form

$$\frac{\partial^2}{\partial\xi'.\partial\eta'} = \frac{\partial^2}{\partial\xi.\partial\eta}. \qquad (a)$$

Now, considering ξ', η' as functions of ξ, η, *without* assuming their linearity, we have

$$\frac{\partial}{\partial\xi} = \frac{\partial\xi'}{\partial\xi}\cdot\frac{\partial}{\partial\xi'} + \frac{\partial\eta'}{\partial\xi}\cdot\frac{\partial}{\partial\eta'}$$

and

$$\frac{\partial^2}{\partial\xi\,\partial\eta} = \frac{\partial\xi'}{\partial\xi}\frac{\partial\xi'}{\partial\eta}\cdot\frac{\partial^2}{\partial\xi'^2} + \frac{\partial\eta'}{\partial\xi}\frac{\partial\eta'}{\partial\eta}\cdot\frac{\partial^2}{\partial\eta'^2} + \left(\frac{\partial\xi'}{\partial\eta}\frac{\partial\eta'}{\partial\xi} + \frac{\partial\xi'}{\partial\xi}\frac{\partial\eta'}{\partial\eta}\right)\frac{\partial^2}{\partial\xi'\partial\eta'}$$

$$+\frac{\partial^2\xi'}{\partial\xi\,\partial\eta}\cdot\frac{\partial}{\partial\xi'} + \frac{\partial^2\eta'}{\partial\xi\,\partial\eta}\cdot\frac{\partial}{\partial\eta'}.$$

Thus, by (a),

$$\frac{\partial^2\xi'}{\partial\xi\,\partial\eta} = 0, \qquad \frac{\partial^2\eta'}{\partial\xi\,\partial\eta} = 0,$$

$$\frac{\partial\xi'}{\partial\xi}\frac{\partial\xi'}{\partial\eta} = 0, \qquad \frac{\partial\eta'}{\partial\xi}\frac{\partial\eta'}{\partial\eta} = 0,$$

$$\frac{\partial\xi'}{\partial\eta}\frac{\partial\eta'}{\partial\xi} + \frac{\partial\xi'}{\partial\xi}\frac{\partial\eta'}{\partial\eta} = 1.$$

To satisfy the third of these conditions, put

$$\frac{\partial\xi'}{\partial\eta} = 0\,;$$

then the fifth will become

$$\frac{\partial\xi'}{\partial\xi}\frac{\partial\eta'}{\partial\eta} = 1,$$

so that the only possibility of fulfilling the fourth condition consists in taking

$$\frac{\partial\eta'}{\partial\xi} = 0.$$

Thus,

$$\xi' = \xi'(\xi), \quad \eta' = \eta'(\eta).$$

Hereby the first and second of the above conditions are identically satisfied, and the fifth becomes

$$\frac{d\xi'}{d\xi} \cdot \frac{d\eta'}{d\eta} = 1. \tag{b}$$

[An alternative solution would be $\partial\xi'/\partial\xi = 0$, and $\partial\eta'/\partial\eta = 0$, *i.e.* $\xi' = \xi'(\eta)$, $\eta' = \eta'(\xi)$, with $(d\xi'/d\eta).(d\eta'/d\xi) = 1$; but this may easily be shown to lead substantially to the same final result as the above one.] Now, for

$$x = vt = c\beta t,$$

we require $x' = 0$, *i.e.*

$$\xi'[ct(\beta - 1)] + \eta'[ct(\beta + 1)] = 0,$$

for every t; hence, differentiating with respect to t, and supposing v *constant*,

$$(1 - \beta)\left(\frac{d\xi'}{d\xi}\right) = (1 + \beta)\left(\frac{d\eta'}{d\eta}\right),$$

and, by (b),

$$\left(\frac{d\xi'}{d\xi}\right) = \sqrt{\frac{1 + \beta}{1 - \beta}}, \quad \left(\frac{d\eta'}{d\eta}\right) = \sqrt{\frac{1 - \beta}{1 + \beta}},$$

where both square roots are to be taken with the same sign, namely the positive (since $\xi' = \xi$, etc., for $\beta = 0$). Here ( ), in the differential coefficients, means 'for $x = vt$'; but since ξ', η' depend only on ξ, η respectively, these formulae are valid for any arguments. Hence, integrating, and remembering that for $x = t = 0$, *i.e.* for $\xi = \eta = 0$, we require $\xi' = \eta' = 0$,

$$\xi' = \sqrt{\frac{1 + \beta}{1 - \beta}}\,\xi; \quad \eta' = \sqrt{\frac{1 - \beta}{1 + \beta}}\,\eta. \tag{c}$$

This intermediate form is worth notice, since it shows at once that

$$\xi'\eta' = \xi\eta,$$

i.e. $x'^2 - c^2t'^2 = x^2 - c^2t^2$.

Introducing again the values of ξ, etc., in terms of x, etc., (c) are readily seen to be identical with the required formulae

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{v}{c^2}x\right).$$

CHAPTER V.

VARIOUS REPRESENTATIONS OF THE LORENTZ TRANSFORMATION.

PASSING now to consider the various expressions of the Lorentz transformation, which was seen to be fundamental for the whole theory of Relativity, let us first of all deprive the x-axis of its (formal) privilege and write (10), Chap. IV., symmetrically in x, y, z, or, using vectors, avoid splitting into Cartesians altogether. This is done in a moment. In fact, remembering that our axis of x was longitudinal, and those of y, z transversal, and calling $\mathbf{r}$ the vector drawn from O to any point in S, and $\mathbf{r}'$ its S'-correspondent, we can write the first of (10),

$$(\mathbf{r}'\mathbf{i}) = \gamma[(\mathbf{r}\mathbf{i}) - vt],$$

where $\mathbf{i}$ is the unit of $\mathbf{v}$, similarly the second and third,

$$\mathbf{r}' - (\mathbf{r}'\mathbf{i})\mathbf{i} = \mathbf{r} - (\mathbf{r}\mathbf{i})\mathbf{i},$$

and, finally, the last of (10),

$$t' = \gamma\left[t - \frac{v}{c^2}(\mathbf{r}\mathbf{i})\right] = \gamma\left[t - \frac{1}{c^2}(\mathbf{r}\mathbf{v})\right].$$

To obtain the full vector $\mathbf{r}'$ combine its transversal and longitudinal parts, and to get rid of the new letter $\mathbf{i}$, write $(\mathbf{r}\mathbf{i})\mathbf{i} = (\mathbf{r}\mathbf{v})\mathbf{v}/v^2$. Thus, the concise vectorial form of the Lorentz transformation, exhibiting its independence of the choice of coordinate axes, will be

$$\left.\begin{aligned} \mathbf{r}' &= \mathbf{r} + \left[\frac{\gamma - 1}{v^2}(\mathbf{v}\mathbf{r}) - \gamma t\right]\mathbf{v} \\ t' &= \gamma\left[t - \frac{1}{c^2}(\mathbf{v}\mathbf{r})\right]. \end{aligned}\right\} \qquad (1)$$

Here $\mathbf{v}$ is the velocity of S' relative to S, and $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, $\beta = v/c$, as before.

To suit the non-vectorial reader we may again split (1) into Cartesians. But in doing so, let us this time take *any* set of mutually perpendicular axes x, y, z, for S, which are also to be the axes of $x', y', z', x'', y'', z''$, etc., for all other systems S', S'', etc., moving uniformly with respect to one another. Call v_x, v_y, v_z the components of $\mathbf{v}$ taken along these universal, but quite arbitrary, axes. Then, projecting the first of (1) upon these axes and re-writing the second of (1), the required symmetrical form will follow, viz.

$$\left.\begin{aligned} x' &= x + \left[\frac{\gamma - 1}{v^2}(\mathbf{rv}) - \gamma t\right] v_x \\ y' &= y + \left[\frac{\gamma - 1}{v^2}(\mathbf{rv}) - \gamma t\right] v_y \\ z' &= z + \left[\frac{\gamma - 1}{v^2}(\mathbf{rv}) - \gamma t\right] v_z \\ t' &= \gamma\left[t - \frac{1}{c^2}(\mathbf{rv})\right], \end{aligned}\right\} \qquad (1a)$$

where $(\mathbf{rv})$ may be looked at as an abbreviation for $xv_x + yv_y + zv_z$. The inverse transformation is obtained by transferring the dashes from x', y', z', t' to x, y, z, t, and by changing the sign of $\mathbf{v}$, that is of v_x, v_y, v_z.

On the other hand, to condense the vectorial form (1) still a little more, observe that $\mathbf{r}$ enters into the first of (1) by the expression $\mathbf{r} + \frac{\gamma - 1}{v^2}\mathbf{v}(\mathbf{vr})$ only. Introduce therefore the *linear vector operator*

$$\epsilon = 1 + \frac{\gamma - 1}{v^2}\mathbf{v}(\mathbf{v} \quad . \qquad (2)$$

Then the Lorentz transformation will be expressed by

$$\left.\begin{aligned} \mathbf{r}' &= \epsilon\mathbf{r} - \mathbf{v}\gamma t \\ t' &= \gamma\left[t - \frac{1}{c^2}(\mathbf{rv})\right]. \end{aligned}\right\} \qquad (1b)$$

Write again, for a moment, $\mathbf{v}/v = \mathbf{i}$, and let $\mathbf{j}$, $\mathbf{k}$ be a pair of unit vectors normal to one another and to $\mathbf{v}$. Then (2) may be written $\epsilon = \gamma\mathbf{i}(\mathbf{i} + 1 - \mathbf{i}(\mathbf{i}$, or, 1 being the 'idemfactor,' *i.e.* $\mathbf{i}(\mathbf{i} + \mathbf{j}(\mathbf{j} + \mathbf{k}(\mathbf{k}$,

$$\epsilon = \gamma\mathbf{i}(\mathbf{i} + \mathbf{j}(\mathbf{j} + \mathbf{k}(\mathbf{k} \quad .$$

This is called a **dyadic.*** Considered as an operator it is a *symmetrical* linear vector operator, so that if **A**, **B** be any pair of vectors

$$(\mathbf{A}\,.\,\epsilon\mathbf{B})=(\mathbf{B}\,.\,\epsilon\mathbf{A}). \tag{3}$$

But the operator ϵ may be described most immediately by calling it a **longitudinal stretcher**, since it stretches or magnifies γ times any longitudinal vector, *i.e.* any vector parallel to $\mathbf{v}$, and leaves unchanged any transversal vector. According to the usual terminology, γ would be *the ratio* of this stretcher.

Observe that $\mathbf{v}$ enters into ϵ through γ only, *i.e.* quadratically. Thus, the inverse transformation will be

$$\left.\begin{aligned} \mathbf{r}&=\epsilon\mathbf{r}'+\mathbf{v}\gamma t' \\ t&=\gamma\left[t'+\frac{1}{c}(\mathbf{v}\mathbf{r}')\right]. \end{aligned}\right\} \tag{1'b}$$

c^{2} ?

The above form of the Lorentz transformation, involving (one vectorial parameter $\mathbf{v}$ or) *three scalar parameters* v_x, v_y, v_z, is especially useful when there are more than two systems, S, S', S'', to be considered, and when the velocity of S'' relative to S' is *not* parallel to that of S' relative to S.

But before proceeding further let us yet dwell a little more upon the properties of the sub-group contained in (1*b*), which involves *one* scalar parameter only, and which covers the particular case of *parallel* velocities. This case is especially interesting and instructive as illustrating a fundamental theorem of Lie's theory of groups of transformations† and as preparing the way for a subsequent form of the Lorentz transformation, adopted for illustrative purposes by Minkowski.

Measuring x, and x', along the direction of motion of S' relative to S, write again, as in the last chapter,

$$x'=\gamma(x-vt),\quad t'=\gamma\left(t-\frac{v}{c^2}x\right),\quad y'=y,\quad z'=z,\ddagger$$

*Cf. for instance my *Vectorial Mechanics*, London, Macmillan & Co., 1913, p. 97. The dots used there as separators are here replaced by (. Thus $\epsilon\mathbf{r}$ means $\gamma\mathbf{i}(\mathbf{ir})+\mathbf{j}(\mathbf{jr})+\mathbf{k}(\mathbf{kr})=\gamma\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$.

† *Theorem* 3 in Vol. I. of S. Lie's *Theorie der Transformationsgruppen*, Leipzig, 1888, p. 33. See also the whole of 'Kapitel 3. *Ein*gliedrige Gruppen und infinitesimale Transformationen,' *Ibidem*, p. 45.

‡ That these transformations form a *group*, and that therefore Lie's theorem must be applicable to them, is easily seen. In fact, if v_1 is the velocity of S'

and differentiate x', y', z', t' with respect to the parameter v. Then, denoting $d\gamma/dv$ by $\dot{\gamma}$,

$$\frac{dx'}{dv}=\frac{\dot{\gamma}}{\gamma}x'-\gamma t;\quad \frac{dt'}{dv}=\dot{\gamma}\left(t-\frac{v}{c^2}x\right)-\frac{\gamma}{c^2}x,$$

and using the inverse transformation $x=\gamma(x'+vt')$, etc.,

$$\frac{dx'}{dv}=\left(\frac{\dot{\gamma}}{\gamma}-\frac{\gamma^2 v}{c^2}\right)x'-\gamma^2 t';\quad \frac{dy'}{dv}=0;\quad \frac{dz'}{dv}=0, \tag{4}$$

$$\frac{dt'}{dv}=\left(\frac{\dot{\gamma}}{\gamma}-\frac{\gamma^2 v}{c^2}\right)t'-\frac{\gamma^2}{c^2}x'. \tag{5}$$

To see that this is precisely the form corresponding to Lie's theorem, which, writing a instead of v, and x_i' $(i=1, 2, 3, 4)$ for x', y', z', t', would be

$$\frac{dx_i'}{da}=\psi_i(a)\,.\,\xi_i(x_1', x_2', x_3', x_4'), \tag{6}$$

we have to remember only that $\gamma^2=(1-\beta^2)^{-1}$, $\beta=v/c$, so that

$$\dot{\gamma}=\frac{1}{c}\beta\gamma^3,$$

and consequently

$$\dot{\gamma}/\gamma-\gamma^2 v/c^2=0,$$

relative to S, and v_2 that of S'' relative to S' (v_2 being taken from the S'-point ot view and v_1 from the S-standpoint), then we have

$$x'=\gamma_1(x-v_1 t),\quad t'=\gamma_1\left(t-\frac{v_1}{c^2}x\right),\quad y'=y,\quad z'=z$$

and

$$x''=\gamma_2(x'-v_2 t'),\quad t''=\gamma_2\left(t'-\frac{v_2}{c^2}x'\right),\quad y''=y',\quad z''=z',$$

and substituting the first in the second, we obtain at once

$$x''=\gamma(x-vt),\quad t''=\gamma\left(t-\frac{v}{c^2}x\right),\quad y''=y,\quad z''-z,$$

which is again a Lorentz transformation like each of the above ones, namely with the parameter (velocity of S'' relative to S)

$$v=\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

This formula embodies the simplest case of Einstein's 'addition-theorem' of velocities, which will occupy our attention in the next chapter.

identically. Thus, the differential equations (4), (5), with the omission of the obvious $dy'/dv = dz'/dv = 0$, become at once

$$\frac{dx'}{dv} = -\gamma^2 t'; \quad \frac{dt'}{dv} = -\frac{\gamma^2}{c} x', \tag{7}$$

or, writing

$$l' = \iota c t', \quad \text{and similarly} \quad l = \iota c t, \tag{8}$$

where $\iota = \sqrt{-1}$,

$$\left.\begin{aligned} \frac{dx'}{dv} &= \iota \frac{\gamma^2}{c} l' \\ \frac{dl'}{dv} &= -\iota \frac{\gamma^2}{c} x'. \end{aligned}\right\} \tag{9}$$

Here, the coefficient on the right side being in both equations the same known function of v, the idea easily suggests itself to introduce instead of v the new parameter

$$\omega = \frac{\iota}{c} \int_0^v \gamma^2 dv = \iota \int_0^\beta \frac{d\beta}{1 - \beta^2},$$

i.e.

$$\omega = \arctan(\iota\beta). \tag{10}$$

With this new variable the above equations become

$$\frac{dx'}{d\omega} = l'; \quad \frac{dl'}{d\omega} = -x'. \tag{9a}$$

Using the well-known general integral of these simple equations and remembering that for $\beta = 0$ (*i.e.* for $\omega = 0$) $x' = x$, $l' = l$, we obtain the remarkable expression of the Lorentz transformation:

$$\left.\begin{aligned} x' &= x \cos \omega + l \sin \omega; \quad y' = y; \quad z' = z \\ l' &= l \cos \omega - x \sin \omega, \end{aligned}\right\} \tag{11}$$

which was first given by Minkowski, who made it his starting point.*

Thus, the Lorentz transformation may be described as a *rotation, in the four-dimensional space* x, y, z, l, *through an imaginary angle* ω *in the plane* x, l, *or 'round the plane'* y, z.

* H. Minkowski, 'Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern,' *Göttinger Nachrichten*, 1907; reprinted in 'Zwei Abhandlungen über die Grundgleichungen der Elektrodynamik,' Teubner, Leipzig, 1910, p. 10.

That the transformation in question is a *pure* rotation, *i.e.* without change of 'length,' $(x^2+y^2+z^2+l^2)^{\frac{1}{2}}$, is best seen from (9*a*), which give at once

$$\frac{d}{d\omega}(x'^2+l'^2)=0,$$

showing thus the invariance, already noticed, of x^2+l^2, and consequently also of $x^2+y^2+z^2+l^2$. Notice that the above rotation ω is an imaginary Euclidean rotation in x, y, z, l, or, which is the same thing, a real non-Euclidean (Lobatchewskyan) rotation in the space x, y, z, ct through an angle ψ connected with ω by

$$\tan\omega=\iota\tan\psi. \tag{12}$$

We shall soon have an opportunity to return to this real angle, which, according to (10), is defined by

$$\tan\psi=\beta. \tag{13}$$

Let again $\mathbf{v}_1$ be the velocity of S' relative to S, and $\mathbf{v}_2$ that of S'' relative to S', the former from the S- and the latter from the S'-point of view. Then, if $\mathbf{v}_1$ and $\mathbf{v}_2$ be parallel to and, say, concurrent with one another, the corresponding rotations are

$$\omega_1=\text{arc}\tan(\iota\beta_1)$$

round a certain plane, in the four-dimensional space x, y, z, l, and

$$\omega_2=\text{arc}\tan(\iota\beta_2)$$

round the same plane. (In three dimensions the rotation is round an axis, or line, in four 'round a plane,' *i.e.* leaving fixed a whole plane instead of a line.) Thus, the resultant rotation, corresponding to the passage from the S- to the S''-variables, will be

$$\omega=\omega_1+\omega_2. \tag{14}$$

Not the velocities themselves are added but the corresponding angles of rotation.

To verify the last formula, call $v=c\beta$ the resultant velocity, corresponding to ω. Then

$$\iota\beta=\tan\omega=\tan(\omega_1+\omega_2)=\iota\,\frac{\beta_1+\beta_2}{1+\beta_1\beta_2},$$

or

$$v=\frac{v_1+v_2}{1+\dfrac{v_1v_2}{c^2}}.$$

Now, this is but a particular case (cf. footnote on pp. 125-6) of Einstein's general formula for the composition of velocities, to be fully considered later on.

Since the sub-group under consideration contains the identical transformation, namely for $v=0$ or $\omega=0$, it must be possible, according to Lie's Theorem 6 (*loc. cit.* p. 49), to represent it as a '*group of translations*,' *i.e.* by

$$\phi_1'=\phi_1; \quad \phi_2'=\phi_2; \quad \phi_3'=\phi_3; \quad \phi_4'=\phi_4-\omega.$$

In fact, by (9*a*) we have the simultaneous system

$$\frac{dx'}{l'}=-\frac{dl'}{x'}=d\omega; \quad dy'=dz'=0,$$

with the initial conditions $x'=x$, $y'=y$, $z'=z$, $l'=l$, for $\omega=0$. Whence

$$x'^2+l'^2=x^2+l^2=\phi_1^2, \text{ say,}$$

and

$$\frac{dl'}{\sqrt{\phi_1^2-l'^2}}=-d\omega.$$

Thus, we have only to write

$$\phi_1=(x^2+l^2)^{\frac{1}{2}}; \quad \phi_2=y; \quad \phi_3=z; \quad \phi_4=\arc\sin\frac{l}{\sqrt{x^2+l^2}}, \tag{15}$$

and the Lorentz transformation will assume the required canonic form

$$\phi_\iota'=\phi_\iota \ (\iota=1,2,3); \quad \phi_4'=\phi_4-\omega. \tag{16}$$

The interpretation of this simple result, and especially that of the meaning of ϕ_4, is left to the reader.

We shall now pass to a remarkable and instructive graphic representation of the Lorentz transformation, due to Minkowski.*

Minkowski calls a space-point at an instant of time, *i.e.* the whole tetrad of values x, y, z, t, a **world-point** (Weltpunkt), and the four-dimensional manifold of all possible systems of values x, y, z, t the **world** (die Welt). Thus, a point of the world represents a material, or, in Minkowski's terminology, a 'substantial' particle at a certain instant. Suppose that the particle can be recognized and watched

*H. Minkowski, 'Raum und Zeit,' lecture delivered during the meeting of the 'Naturforscherversammlung' at Cologne, 1908, *Physik. Zeitschrift*, Vol. X., p. 104, 1909, reprinted, with a preface by A. Gutzmer, by B. G. Teubner, Leipzig and Berlin, 1909.

during its whole history. Then a one-dimensional continuum, contained in the four-dimensional world, may be constructed, whose element has the components

$$dx,\ dy,\ dz,\ dt$$

along the space- and time-axes, and which represents the history of the particle. This line, whose points may be uniquely referred to the parameter t, say, from $-\infty$ to $+\infty$, is called a **world-line** (Weltlinie). Thus the whole world would consist of a maze of such world-lines, and the physical laws would find 'their most perfect expression in the mutual relations obtaining between these world-lines.' This, of course, can be only an ideal task, and in putting it before the eyes of physicists and mathematicians, Minkowski, no doubt, was very well aware how far we are from its accomplishment.

If instead of a particle or substantial point we have a body of finite space-extension, then drawing through each of its points a world-line, we shall obtain a tubular portion of the four-dimensional world, which may be called a **world-tube.** In his previous paper, of 1907,* Minkowski calls it a **space-time filament.** The utility of the conception of a space-time filament or tube in mechanical problems and those concerned with the motion of electrons is obvious.

The world-line of a particle will in general be curvilinear, *e.g.* for any non-uniform motion, whether the particle's path or orbit in ordinary space be curvilinear or its velocity be changing in absolute value. But if the particle is moving uniformly, with respect to a given system S (x, y, z, t), then its world-line will be a straight line, which means only that the corresponding equations obtaining between the four variables will be linear. In particular, if the particle is at rest in S, then its world-line will coincide with the t-axis, this axis, as also the axes of x, y, z, being considered as straight lines in the four-dimensional world.

The complete representation cannot of course be given, either by a plane drawing or by a three-dimensional model.† But this is no serious objection against Minkowski's method. For, first of all, it

* *Grundgleichungen für die elektromagnetischen Vorgänge*, p. 47.

† For a remarkable attempt to obtain a geometrical image of Minkowski's world by means of systems of spheres see a paper by H. E. Timerding in *Jahresbericht der deutschen Math. Vereinigung*, Vol. XXI. 1913, p. 274.

is very advantageous, especially for the trained geometer of our days, even merely to think and to speak about these relations in terms of four-dimensional geometry. And then we can help ourselves by taking various sections of the four-dimensional world, by constructing three-dimensional models (x, y, t, or y, z, t, etc.) or, still better, plane drawings in t and one of the space-axes.

It is such a graphic representation that we are offered in Minkowski's inspired lecture.

Let B_1OB (Fig. 12) be the axis of ct, and A_1OA that of x.* Draw the straight line L_1OL bisecting the right angle AOB. This line would represent the world-line of a particle moving uniformly,

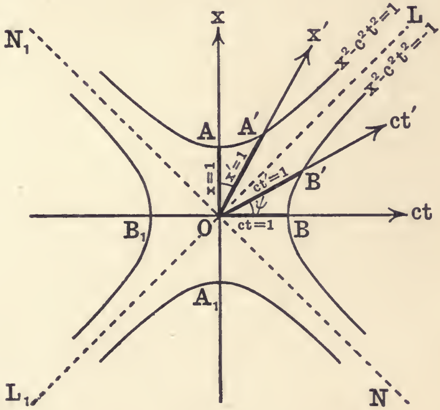

FIG. 12.

along the axis of x, with the velocity of light c. Now, according to one of the assumptions of the theory of relativity, the velocity v of any particle is always smaller than c, or at least does not exceed c. Consequently no world-line will be steeper than, or even as steep as, L_1OL or N_1ON. Every world-line passing through O, *i.e.* belonging to a particle for which $x=0$ at the instant $t=0$, is entirely confined to the region consisting of LON and L_1ON_1. For, to penetrate into LON_1 or NOL_1, the particle would have to move, at least during a certain part of its wandering, with a hypervelocity.

*The plotting of x and the time against one another has, of course, nothing novel about it. It is familiar to everybody from elementary text-books on mechanics by the name of a 'displacement curve.' But none the less its application to relativistic connexions has been a happy idea.

Let OB' be a world-line representing a particle in uniform motion with velocity $v = c\beta$. Then

$$\tan\psi = \beta,$$

where ψ is the angle BOB'. Notice that our previous angle ω, endowed with the remarkable additive property with regard to the composition of parallel velocities, is connected with this real angle BOB' by $\tan\omega = \iota\tan\psi$. By what has been said above, the absolute value of the trigonometric tangent of this angle is smaller than unity,

$$|\tan\psi| < 1.$$

Now, to obtain a representation of the Lorentz transformation from $S(x, t)$ to the system $S'(x', t')$ attached to our uniformly moving particle, draw the hyperbola

$$x^2 - c^2t^2 = -1 \tag{17}$$

and the conjugate hyperbola

$$x^2 - c^2t^2 = 1, \tag{18}$$

of which the previous L_1OL, N_1ON, given by

$$x^2 - c^2t^2 = 0, \tag{19}$$

is the common pair of asymptotes.

In order to represent the particle as being at rest, *i.e.* in order to pass from S to S', take OB', instead of OB, as the new axis of time, that is to say of ct', and as the axis of x' a straight line OA', such that

$$LOB' = LOA'$$

or

$$AOA' = BOB' = \psi,$$

and, instead of OA and OB, the segments OA' and OB' as the units of x' and ct', as explained in Fig. 12. The obvious proof that this is equivalent to the Lorentz transformation $x' = \gamma(x - vt)$, $t' = \gamma(t - vx/c^2)$, is left to the reader. Notice further that, by construction, OA' and OB' are *conjugate* semi-diameters of the hyperbola $x^2 - c^2t^2 = -1$, as were also OA and OB.

Thus, the Lorentz transformation consists in passing from one to another pair of conjugate semi-diameters of the hyperbola $x^2 - c^2t^2 = -1$ and in taking their lengths as the new units for x and ct.*

*Here, as before, x, that is to say x for S as well as the new x' for S', is the coordinate measured along $\mathbf{v}$, the velocity of S' with respect to S.

The new x- and t-axes are obtained by turning each of the old ones, towards or away from the asymptote OL, through the angle

$$\psi = \arc\tan\beta,$$

not exceeding $45°$.

Since $x^2 - c^2t^2$ is invariant with respect to the Lorentz transformation, the asymptotes L_1OL and N_1ON and the hyperbolae are fixed, *i.e.* remain always the same no matter whether x, t or x', t' or x'', t'', etc., are chosen as variables. The same property belongs of course to the whole system of hyperbolae

$$x^2 - c^2t^2 = -\kappa^2$$

and of the conjugate hyperbolae

$$x^2 - c^2t^2 = \kappa^2,$$

where κ is any real number. The asymptotes may be considered as a particular, limiting case of these curves, corresponding to $\kappa = 0$.

The reader is recommended to compare the case under consideration with that of an ordinary rotation of a plane, say x, y, in itself, when $x^2 + y^2 = \kappa^2$ is invariant, giving circles, instead of hyperbolae, as permanent paths of the points of the plane, and a single fixed point $\kappa = 0$ instead of a pair of straight lines. In connexion with this remark *hyperbolic* functions may conveniently be introduced, to replace the ordinary sine and cosine. Writing

$$\tan\psi = \tanh a, \tag{20}$$

Fig. 12 will easily lead to the formulae

$$\left.\begin{aligned} x' &= x\cosh a - ct\sinh a \\ ct' &= ct\cosh a - x\sinh a, \end{aligned}\right\} \tag{21}$$

which agree with (11), since, by (20) and (12),

$$\omega = \iota a. \tag{22}$$

Remember that, by the definition of the hyperbolic functions,

$$\sinh a = -\iota\sin(\iota a),$$

$$\cosh a = \cos(\iota a).$$

Notice that, the region of OB' being LON,* a time-axis can be

* For positive values of t, and L_1ON_1 for negative values of t, and similarly as regards LON_1 when the x-axis is in question.

drawn from O through any world-point situated in this region, that is to say, through any point for which

$$x^2 - c^2t^2 < 0. \tag{23a}$$

Similarly, the region of OA' being LON_1, an x-axis can be drawn through any world-point for which

$$x^2 - c^2t^2 > 0, \tag{23b}$$

so that *any of such world-points can be made simultaneous with O.* This is an eminently characteristic feature of the new doctrine as distinguished from the old system of physics in which simultaneity was an absolute property of events, independent of our choice of a standpoint. It is plainly an immediate consequence of the reform of the concept of simultaneity introduced by Einstein. Pairs of events are or are not simultaneous according to the choice of our standpoint, *i.e.* of one out of an infinity of legitimate systems S, S', etc., in exactly the same way as pairs of space-points have or have not equal values of x and y (or y and z, or z and x) according to our choice of the coordinate-planes. There is thus far an intrinsic similarity, a kind of coordinateness, between space and time, or as the Time Traveller, in a wonderful anticipation of Mr. Wells, puts it: '*There is no difference between Time and Space except that our consciousness moves along it.*' *

The process of passing the time-axis through a world-point corresponding to a given particle, since it brings it to rest, is often referred to as *transforming that particle to rest.*† In view of the above property, a vector ('world-vector,' to be treated fully further on) in the plane x, ct, drawn from O to any point of the region limited by LON, or L_1ON_1, satisfying the condition (23*a*), may be called a **time-like vector**, and a vector drawn from O towards any

* H. G. Wells: *The Time Machine*, 1898 (Tauchnitz edition), p. 13. It is interesting to remark that even the forms used by Minkowski to express these ideas, as 'Three-dimensional geometry becoming a chapter of the four-dimensional physics,' are anticipated in Mr. Wells' fantastic novel. Here is another sample (*loc. cit.* p. 14), illustrative of what is now called a world-tube: 'For instance, here is a portrait [or, say, a statue] of a man at eight years old, another at fifteen, another at seventeen, another at twenty-one, and so on. All these are evidently sections, as it were, Three-Dimensional representations of his Four-Dimensioned being, which is a fixed and unalterable thing.' Thus, Mr. Wells seems to perceive clearly the absoluteness, as it were, of the world-tube and the relativity of its various sections.

† In German, 'Auf Ruhe transformieren.'

point of the remaining region, LON_1 or L_1ON, *i.e.* satisfying (23*b*), a **space-like vector.*** On the border between these two classes of vectors we have **singular vectors,** drawn from the origin to any point of the asymptotes, *i.e.* coinciding, in this bi-dimensional case, in fact, with parts of the asymptotes, and characterized by $x^2 - c^2t^2 = 0$. For $t > 0$ the world-point at the end of a singular vector represents a particle when it just receives a light-signal from O, that is to say, a signal started at $x = 0$ at the instant $t = 0$. Similarly, for $t < 0$, the end-point of a singular vector represents a particle just at the instant when it sends a light-flash which arrives at $x = 0$ at the instant $t = 0$. Or, as Minkowski puts it, L_1ON_1 consists of all the world-points that *send light towards O*, and LON of all those that *receive light from O.*

Notice that $x = \pm ct$, if transformed, gives $x' = \pm ct'$, which follows from the invariance of $x^2 - c^2t^2$ (together with the requirement $x' = x$, $t' = t$ for $v = 0$), and is only a verification of the assumption, made at the outset, that the velocity of light in empty space is the same for all legitimate systems of reference. In this case both x and t are reduced by the Lorentz transformation in the same ratio. In fact, substituting $x = ct$ in $x' = \gamma(x - vt)$, $t' = \gamma(t - vx/c^2)$, we obtain

$$x' : x = t' : t = (1 - \beta)^{\frac{1}{2}}(1 + \beta)^{-\frac{1}{2}}.$$

Thus far we have considered besides t one independent variable only, the space coordinate x. Accordingly, any world-line, traced in that bi-dimensional diagram, has been the representation of a particle, or, in the limiting case, of a flash of light travelling along a straight line, the x-axis. Now, bring in the coordinate y. Then the resulting three-dimensional diagram or model will be appropriate to represent the motion of a particle, or the propagation of light, in a plane, the plane of x, y. Return to Fig. 12, and imagine the axis of y to be drawn through O perpendicularly to the paper. To obtain the required representation, we have only to spin the two hyperbolae of Fig. 12 and their asymptotes round B_1OB as axis. The two branches of the hyperbola (17) will generate a *hyperboloid of revolution of two sheets*

$$x^2 + y^2 - c^2t^2 = -1, \tag{24}$$

and the two branches of the hyperbola (18), exchanging rôles after

* If we are to translate thus the names used by Minkowski: *zeitartiger* and *raumartiger Vektor* respectively.

a rotation through 180°, will give rise to a *hyperboloid of revolution of one sheet*

$$x^2+y^2-c^2t^2=1, \tag{25}$$

which will be cut by the y-axis in a pair of points, say, C and C_1, one above and the other below the paper, while the asymptotic lines will generate a right cone

$$x^2+y^2+z^2-c^2t^2=0, \tag{26}$$

the *asymptotic cone* of the hyperboloids. As regards this conic surface, let us distinguish its two parts L_1ON_1 and LON (revolved), corresponding to negative and positive times respectively, and let us call the first the **fore-cone** and the second the **aft-cone** of O.* The fore-cone consists of all world-points, out of those under consideration, which 'send light' towards O, and the aft-cone of all those which 'receive light' from O. Any vector drawn from O to a world-point contained within the fore- or aft-cone will be a time-like vector, and vectors drawn from O to any point of the remaining region of the world, outside the cones, will be space-like vectors.

Now, let $\mathbf{v}$ be the ordinary vector-velocity of a particle in uniform motion, and let it have any direction whatever in the plane of x, y. Then the world-line of this particle will be a straight line passing through O in the plane $\mathbf{v}$, OB, and including with OB, the original time-axis, the angle

$$\psi=\operatorname{arc}\tan\beta,$$

where $\beta=v/c$. To transform the particle to rest, take this world-line as the axis of ct', and to obtain at the same time the new coordinates x', y' turn the old plane xy through the angle ψ round an axis passing through O and perpendicular to both $\mathbf{v}$ and OB. For the moment, call the coordinates measured in the xy-plane, along $\mathbf{v}$ and perpendicularly to it, ξ and η respectively. Then the turning round of that plane from its original position ($t=0$) will amount to writing

$$ct=\xi\,.\tan\psi=\beta\xi.$$

On the other hand, we have

$$x^2+y^2=\xi^2+\eta^2$$

for any point of the plane xy, so that (25) will become

$$\xi^2+\eta^2-c^2t^2=1.$$

* Minkowski, '*Vorkegel*' and '*Nachkegel*.'

The intersection of the new plane, $x'y'$, with the surface (25), will, therefore, be given by

$$(1-\beta^2)\xi^2+\eta^2=1.$$

Now, $\beta^2<1$. Thus the $x'y'$ plane will cut the one-sheeted hyperboloid in an *ellipse*. To complete the Lorentz transformation we have only to take the semi-diameters of this ellipse as *the new units of length* measured from the origin along any direction in the $x'y'$ plane. The major principal axis of this metric ellipse will be contained in the plane **v**, OB, and the other axis will be normal to it. This ellipse of our graphical representation will, of course, in the new units of length, be a circle, *i.e.* $x'^2+y'^2=1$. So also did the old plane of coordinates (xy) cut the one-sheeted hyperboloid in a circle $x^2+y^2=1$. This is seen at once to agree with the invariance of $x^2+y^2-c^2t^2$. We have generally

$$x^2+y^2-c^2t^2=x'^2+y'^2-c^2t'^2,$$

and since the sections under consideration are obtained by putting $t=0$, $t'=0$ respectively, the S-circle

$$x^2+y^2=1$$

has for its S'-correspondent the circle

$$x'^2+y'^2=1.$$

The new unit of time, *i.e.* of ct', is again represented by the segment of the ct'-axis cut off by one of the sheets of the two-sheeted hyperboloid of revolution, *i.e.* by the semi-diameter conjugate to the plane $x'y'$. So also was the old time-axis, OB, conjugate to the old plane of coordinates (xy), and the unit of ct was the semi-diameter OB.

To resume this three-dimensional graphic representation:

> The Lorentz transformation consists in passing from one to another set composed of a time-like semi-diameter and the *conjugate* space-like semi-diameters of the hyperboloid
>
> $$x^2+y^2-c^2t^2=-1,$$
>
> and in taking the lengths of the new semi-diameters as the units for the time (ct') and for the space-coordinates; the units of length being thus given in each case by the semi-diameters of the ellipse cut out from the one-sheeted hyperboloid $x^2+y^2-c^2t^2=1$ by the plane of coordinates.

The new time-axis and the new coordinate-plane are obtained by turning each of the old ones, towards or away from the asymptotic cone round an axis passing through O and perpendicular both to the old time-axis and to the velocity $\mathbf{v}$ of the new system with respect to the old one.

Having gone through all of this, we can now pass to the most general, four-dimensional case. Here, it is true, our imagery fails us. But we can still advantageously avail ourselves of the geometrical language as a guide to, and as a short expression of, the analytical process involved.

Instead of the hyperboloidic surfaces we have now the **two-'sheeted' hyperboloidic space** or, as we may conveniently call it, **the double hyperboloid**

$$r^2 - c^2t^2 \equiv x^2 + y^2 + z^2 - c^2t^2 = -1 \qquad (27)$$

and its conjugate, the **one-'sheeted' hyperboloidic space** or **the single hyperboloid**

$$r^2 - c^2t^2 \equiv x^2 + y^2 + z^2 - c^2t^2 = 1, \qquad (28)$$

with their common **asymptotic conic space**

$$r^2 - c^2t^2 \equiv x^2 + y^2 + z^2 - c^2t^2 = 0, \qquad (29)$$

consisting of the **fore-cone** $t < 0$ and the **aft-cone** $t > 0$, as before, with the only difference that these, like the hyperboloids, are now three-dimensional entities.

The t-axis cuts the double hyperboloid (27) in a pair of points, namely

$$x = y = z = 0, \quad ct = 1$$

and

$$x = y = z = 0, \quad ct = -1.$$

Take the first, contained in the positive 'sheet.' Call it P, so that OP, a semi-diameter of the hyperboloid (27), is the old time-axis, and the length of this semi-diameter is the unit of ct. The space $t = 0$, that is to say, the ordinary space-manifold x, y, z is the three-space conjugate to the semi-diameter OP, just as the xy-plane, in the previous case, was conjugate to OB (Fig. 12). Now, instead of P, take any other point P' of the positive sheet of (27), and consider OP' as the new time-axis and the length of this semi-diameter as the unit of ct'. Turn the xyz-space ($t = 0$) which cut the single hyperboloid (28) in a sphere,

$$x^2 + y^2 + z^2 = 1,$$

round the plane passing through O and perpendicular to $\mathbf{v}$, till this space, or pencil of semi-diameters, becomes conjugate to the semi-diameter OP'. Then it will become the $x'y'z'$-space. This space cuts the single hyperboloid in an ellipsoid (ellipsoidic surface). Take the semi-diameters of this ellipsoid as the new units of length measured from the origin along any direction in the $x'y'z'$-space. Then the Lorentz transformation, from S to S', will be completed, and the new metric surface which, from the S-point of view, is an ellipsoid of revolution will for the S'-standpoint become a sphere,

$$x'^2+y'^2+z'^2=1.$$

So also was the old metric surface, viewed from the old standpoint, a sphere of unit radius. Remember that OP' is time-like, *i.e.* contained within the four-dimensional region bounded by the three-dimensional cone, but otherwise the choice of this axis as a time-axis is free. The possible positions of P' constitute a triple manifold, namely all the points of the positive sheet of (27). Thus, the systems $S'(x', y', z', t')$ equally legitimate with S are ∞^3, as has been repeatedly observed.*

To resume what has just been said with regard to the general, four-dimensional case:

> The Lorentz transformation consists in passing from one (time-like) semi-diameter OP and the pencil of *conjugate* (space-like) semi-diameters of the hyperboloid $r^2-c^2t^2=-1$ to another semi-diameter OP' with its corresponding pencil of conjugate semi-diameters, and in taking the lengths of the new semi-diameters as the units of time (ct') and of space-coordinates; the units of length being thus given in each case by the semi-diameters of the ellipsoid cut out from the hyperboloid $r^2-c^2t^2=1$ by the new space of coordinates.

The property of two lines OP_1 and OP_2 being *conjugate* may be expressed analytically by the equation

$$x_1x_2+y_1y_2+z_1z_2-c^2t_1t_2=0, \tag{30a}$$

where x_1, y_1, z_1, ct_1 and x_2, y_2, z_2, ct_2 are the values of the four

* The simple turning round of x, y, z, leaving $x^2+y^2+z^2$ invariant, being always left out of account. Having once assumed the *isotropy* of space, we have, speaking physically, no need to consider such rotations. And with regard to their mathematical rôle, see Chap. VI.

variables defining the world-points P_1 and P_2 respectively, or, using the ordinary vectors $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$(\mathbf{r}_1\mathbf{r}_2) - c^2 t_1 t_2 = 0, \tag{30b}$$

or finally, writing $l = \iota ct$,

$$(\mathbf{r}_1\mathbf{r}_2) + l_1 l_2 = 0. \tag{30}$$

By an obvious analogy such lines OP_1, OP_2 are also called mutually **perpendicular** or **normal** lines in the world x, y, z, l. Notice that this property of a pair of lines is *invariant* with respect to the Lorentz transformation, *i.e.* that (30) is transformed into

$$(\mathbf{r}_1'\mathbf{r}_2') + l_1' l_2' = 0.$$

In other words, conjugate diameters remain conjugate, independently of the choice of a reference-system. This is obvious, at least in two and three dimensions. More generally, for *any* pair of lines OP_1, OP_2,

$$(\mathbf{r}_1'\mathbf{r}_2') + l_1' l_2' = (\mathbf{r}_1\mathbf{r}_2) + l_1 l_2, \tag{31}$$

as the reader himself may prove, using for instance the form (1*b*) of the Lorentz transformation, and noticing that

$$(\epsilon\mathbf{r}_1 . \epsilon\mathbf{r}_2) - \gamma^2 c^{-2} (\mathbf{r}_1\mathbf{v})(\mathbf{r}_2\mathbf{v}) = (\mathbf{r}_1\mathbf{r}_2)$$

identically. Thus, the invariance of orthogonality is but a particular case of the invariance of

$$(\mathbf{r}_1\mathbf{r}_2) + l_1 l_2.$$

We shall return to the last property later on.

Given the origin O (and *any* world-point can be made the origin), the set of any four values of

$$x,\ y,\ z,\ l,$$

or, more generally, of any four scalar magnitudes

$$w_x,\ w_y,\ w_z,\ s,$$

which are transformed like x, y, z, l respectively, and of which the first three are real and the fourth purely imaginary, defines what is called a **world-vector** or **space-time vector of the first kind*** (Minkowski) or a **four-vector** (Sommerfeld).

* To be distinguished, later on, from those 'of the second kind' or 'six-vectors.'

Thus, if (30) is satisfied, the four-vectors OP_1 and OP_2 are said to be perpendicular to one another. Generally, if

$$(\mathbf{w}_1\mathbf{w}_2)+s_1s_2=0, \tag{32}$$

then $\mathbf{w}_1, s_1$ and $\mathbf{w}_2, s_2$ form a pair of perpendicular four-vectors. Here $\mathbf{w}_1$ is the ordinary or three-vector whose components are w_x, w_y, w_z, and $\mathbf{w}_2$ has a similar meaning, while $(\mathbf{w}_1\mathbf{w}_2)$ is, as before, the ordinary scalar product of $\mathbf{w}_1$, $\mathbf{w}_2$.

Any four-vector drawn from O to a world-point contained within the asymptotic cone, *i.e.* such that $r^2-c^2t^2=r^2+l^2<0$ or, more generally, any four-vector $\mathbf{w}$, s, such that

$$w^2+s^2<0, \tag{33t}$$

is called, as in the two- and three-dimensional cases, a **time-like vector,** while four-vectors satisfying the condition $r^2+l^2>0$ or, generally,

$$w^2+s^2>0, \tag{33s}$$

are called **space-like vectors.**

The reader will easily prove that *if one of a pair of normal four-vectors is time-like, the other is space-like*, or that, in other words, if one is contained within the asymptotic cone, the other is outside it.

Again, as in the above special cases, any vector drawn from O towards a point of the asymptotic cone, whether the fore- or aft-cone, is called a **singular four-vector.** The analytical expression of a singular vector is $r^2-c^2t^2=r^2+l^2=0$, or, generally,

$$w^2+s^2=0. \tag{34}$$

Finally, as in the less-dimensional cases, the aft-cone may be said to consist of all world-points which 'receive light' from O, and the fore-cone of all those that 'send light' towards O.

Fig. 13, which is Fig. 12 redrawn with the omission of the arbitrary axes, and thus contains only what is 'absolute' or independent of the choice of such time- and space-axes, may aid the reader in remembering the meaning of the various names employed in the above representation. This figure is drawn perspectively (in three dimensions, of course), so as to show that the hyperboloids (27) and (28) are hyperboloids of revolution, the former consisting of two disconnected 'sheets' and the latter of one 'sheet.' We may mention further that the world-region contained

within the fore-cone (left) was called by Minkowski **this side of** O and that contained within the aft-cone **that side of** O. Every world-point of the first region is necessarily (independently of the selection of a reference-system) or **essentially earlier**, and every world-point of the second region is **essentially later** than O. Any point of the remaining, cyclical, region of the world, called the **intermediate region**, can be made simultaneous with or earlier or later than O (*i.e.* can be given a value of t = or $<$ or $>$ o) by an appropriate choice of the time-axis, and is therefore essentially neither earlier nor later than O. This region is the domain of all space-like four-

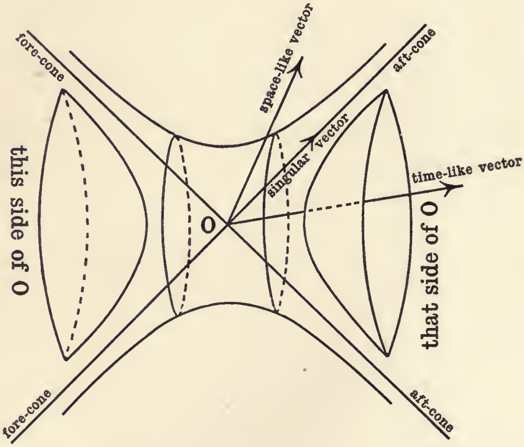

FIG. 13.

vectors which can be drawn from O. Between the time-like and space-like classes of world-vectors are the singular vectors, composing the cones which are three-dimensional entities.

This partitioning of the world and the characteristic properties of the cones are obviously conditioned by the assumption that no particle, or at least, no legitimate system, can ever move (relatively to another one) with a velocity v exceeding that of light in empty space. In classical physics there was no limit whatever to v. The Newtonian transformation follows from the Lorentz transformation by taking ∞ instead of c, or, figuratively, by widening both the cones till they coalesce with one another in a plane, squeezing out the space-like four-vectors and opening the whole world to the time-like vectors. *Any* straight line would, in the Newtonian world,

represent a possible uniform motion of a particle with respect to certain frames of reference.

So much as regards the geometric representation of the Lorentz transformation.

Now for its analytical expression and the methods of dealing with the world-vectors.

Minkowski, though availing himself now and then of the four-dimensional vector language and ideology, made a systematical and extensive use of Cayley's calculus of matrices.* Thus, the fundamental world-vector **r**, l and, more generally, any space-time vector of the first kind **w**, s is considered as a matrix of 1 row and 4 columns, say,

$$X = | \, x, \; y, \; z, \; l \, | \tag{35}$$

and, in general,

$$W = | \, w_x, \; w_y, \; w_z, \; s \, |.$$

The transformed world-vector **r**′, l' will then be another matrix of 1×4 constituents,

$$X' = | \, x', \; y', \; z', \; l' \, |, \tag{35'}$$

which is obtained from X by taking its 'product' into a certain matrix of 4×4 constituents,

$$A = \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14} \\ a_{21}, & a_{22}, & a_{23}, & a_{24} \\ a_{31}, & a_{32}, & a_{33}, & a_{34} \\ a_{41}, & a_{42}, & a_{43}, & a_{44} \end{vmatrix}, \tag{36}$$

and which is written simply

$$X' = XA. \tag{37}$$

Thus, the Lorentz transformation is expressed by the matrix A taken as a *postfactor* of the world-vector to be transformed. This matrix is characterized by the condition that its determinant is $+1$,

$$\det A = 1, \tag{38}$$

and further that all of its constituents containing the index 4 once

*Cf. Minkowski's *Grundgleichungen*, already quoted, §§ 11 *et seq.* Those readers who are not familiar with this branch of mathematics may consult the Note at the end of this chapter, where the definition of different kinds of matrices and some rules of operating with them are given.

only are purely imaginary, while the remaining seven constituents are real and the right lowermost positive:

$$\alpha_{11},\ \alpha_{12}, \ldots \alpha_{33}\ \textit{real}$$

$$\left.\begin{matrix}\alpha_{14},\ \alpha_{24},\ \alpha_{34} \\ \alpha_{41},\ \alpha_{42},\ \alpha_{43}\end{matrix}\right\}\ \textit{purely imaginary}$$

$$\alpha_{44} > 0.$$

The inverse transformation is represented by the *reciprocal* of A, which is at the same time the *transposed* of A, $A^{-1} = \bar{A}$, so that

$$\bar{A}A = A\bar{A} = 1. \tag{39}$$

It is this property that insures the invariance of $r^2 + l^2$. Using $\bar{A}$ and $\bar{X}$, we may write also, instead of (37),

$$\bar{X}' = \bar{A}\bar{X}.$$

The short formula (37) replaces

$$x' = \alpha_{11}x + \alpha_{21}y + \alpha_{31}z + \alpha_{41}l,$$

and three similar equations, with 2, 3, 4 as second indices. If, in particular, the x, y, z-axes are taken along $\mathbf{v}$ and normal to it, and if x', y', z' are, as before, measured along the same directions, then, as we saw,

$$x' = \gamma(x + \iota\beta l)\,; \quad y' = 0\,; \quad z' = 0\,;$$

$$l' = \gamma(l - \iota\beta x).$$

Hence, for this particular choice of coordinate-axes the matrix representing the Lorentz transformation reduces to

$$A = \begin{vmatrix} \gamma, & 0, & 0, & -\iota\beta\gamma \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ \iota\beta\gamma, & 0, & 0, & \gamma \end{vmatrix}. \tag{40}$$

The transposed matrix $\bar{A}$ which represents the inverse Lorentz transformation is obtained from this by simply changing the sign of β, as it should be.

Writing, instead of $x, \ldots l$, the differentiators $\partial/\partial x, \ldots \partial/\partial l$, we obtain a matrix of 1×4 constituents, which Minkowski called **lor,** in honour of Lorentz,

$$\text{lor} = \left| \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial l} \right|. \tag{41}$$

This is the matrix-equivalent of our quaternionic differential operator D, as defined by (13), Chap. II. It can be easily verified that $\partial/\partial x, \ldots \partial/\partial l$ are transformed in exactly the same way as x, y, z, l respectively.* Thus, lor is **covariant**, or equally transformed, with the matrix X representing the standard world-vector, *i.e.*

$$\text{lor}' = \text{lor}\, A. \tag{42}$$

Moreover, it has the same structure as X, its first three constituents (differentiators) being real and the fourth, $\partial/\partial l$, purely imaginary. Thus, lor, though an operator, behaves in every respect like a space-time vector of the first kind.

We cannot stop here to consider the matrix form of space-time vectors of the second kind and their analytical connexion with those of the first kind (although it could be done in a few lines), for the reader does not yet know their relativistic physical significance. Moreover, it is not our purpose to develop fully the matrix method of treating relativistic questions, since we shall avail ourselves chiefly of other methods. But one simple property of products of W-matrices in connexion with the above remarks is worth mentioning here, namely that, if W_1, W_2 are matrices representing a pair of vectors of the first kind ($\mathbf{w}_1, s_1; \mathbf{w}_2, s_2$), the product

$$W_1 \overline{W}_2 = (\mathbf{w}_1 \mathbf{w}_2) + s_1 s_2 \tag{43}$$

is an *invariant*. For by (39), and by the associative property of products of matrices,

$$W_1' \overline{W}_2' = W_1 A \,.\, \bar{A}\, \overline{W}_2 = W_1 \overline{W}_2.$$

* Thus, for instance, measuring x along $\mathbf{v}$, we have

$$x' = \gamma(x + \iota\beta l); \quad y' = y; \quad z' = z; \quad l' = \gamma(l - \iota\beta x),$$

whence $x = \gamma(x' - \iota\beta l')$, etc., and

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \iota\beta\frac{\partial}{\partial l}\right); \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}; \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}; \quad \frac{\partial}{\partial l'} = \gamma\left(\frac{\partial}{\partial l} - \iota\beta\frac{\partial}{\partial x}\right).$$

Thus, the *orthogonality* of two four-vectors, which is an invariant property, is expressed by

$$W_1 \overline{W}_2 = 0.$$

Similarly,

$$\text{lor}' \, \overline{W}' = \text{lor} \, \overline{W},$$

or lor $\overline{W}$ is a relativistic *invariant*. Notice that, similarly to (43),

$$\text{lor} \, \overline{W} = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} + \frac{\partial s}{\partial l},$$

or, using div in its ordinary sense,

$$\text{lor} \, \overline{W} = \text{div} \, \mathbf{w} + \frac{\partial s}{\partial l} \cdot \tag{44}$$

So much as regards Minkowski's matrix-form of the fundamental relativistic connexions.

Sommerfeld, whose aim was to elucidate Minkowski's ideas, replaced his language of matrices by a four-dimensional vector-algebra (and -analysis) which he developed in two very lucid papers,* and which is an obvious generalization of the familiar three-dimensional calculus of vectors. Sommerfeld begins by drawing our attention to the well-known circumstance that in space of three dimensions there are two kinds of vectors to be distinguished, *e.g.* vectors of the 'first kind' or polar, and those of the 'second kind' or axial vectors. A vector of the first kind, such as a translation velocity, is a segment of a straight line having a certain direction (and sense); its components are the projections upon the coordinate-axes. On the other hand, a vector of the second kind, such as angular velocity, is represented by a plane surface of a certain area with a given sense of circulation round its circumference, and its components are the projections of that area upon the coordinate-*planes*. Consequently, the components of a vector of the first kind should be written with single indices, v_x, v_y, v_z, or v_1, v_2, v_3, while those of a vector of the second kind, as, for instance, rotational velocity ω, with double indices, ω_{yz}, ω_{zx}, ω_{xy}, or ω_{23}, ω_{31}, ω_{12}. This discrimination, which in three dimensions is not very important (or at least ceases to be so when, instead of the plane area, a representative line-segment normal to it is introduced), becomes in

* A. Sommerfeld, 'Zur Relativitätstheorie. I. Vierdimensionale Vektoralgebra,' *Ann. der Physik*, Vol. XXXII., 1910, p. 749, and 'II. Vierdimensionale Vektoranalysis,' *Ann. der Physik*, Vol. XXXIII., 1910, p. 649.

Minkowski's four-dimensional world quite essential. For here—argues Sommerfeld—we have

$$\binom{4}{1} = \textit{four}\text{ coordinate axes,}$$

$$x,\ y,\ z,\ l,$$

$$\binom{4}{2} = \textit{six}\text{ coordinate planes,}$$

$$yz,\ zx,\ xy,\ xl,\ yl,\ zl,$$

and

$$\binom{4}{3} = \textit{four}\text{ coordinate spaces,}$$

$$xyz,\ yzl,\ zxl,\ xyl.$$

Accordingly, we have to distinguish in the 'world' between

vectors of **the first kind** having four components, or **four-vectors**; those of **the second kind** having six components, or **six-vectors**;

and, finally,

those of **the third kind**, which again have four components, and can be replaced by their 'supplements,' which are vectors of the first kind.

Consequently, vectors of both the first and the third kind are called by Sommerfeld, summarily, four-vectors.

This classification will be found useful for what is to occupy us later on. But meanwhile we are concerned only with space-time-vectors of the first kind, which we shall simply call four-vectors.

The standard or typical example of such vectors is that drawn from the origin O to any world-point. Call it P.* Then its components would be, according to Sommerfeld's general notation,

$$P_x,\quad P_y,\quad P_z,\quad P_l.$$

These, of which the first three are real and the last imaginary, are simply the previous

$$x,\quad y,\quad z,\quad l.$$

* Sommerfeld does not use any special type of print for his four-vectors, to distinguish them from six-vectors. A certain uniformity of notation was introduced later by Laue, *loc. cit.* But we shall not want very much of it for our subsequent purposes.

What Sommerfeld denotes by $|P|$ and calls the **size** of the vector P, or its **length**, *i.e.* the 'length' of the corresponding four-dimensional straight line, is the positive (or positive imaginary) value of

$$\sqrt{x^2+y^2+z^2+l^2}=\sqrt{x^2+y^2+z^2-c^2t^2},$$

or of $\sqrt{r^2-c^2t^2}$.

The length of this, and of every other, four-vector is invariant with respect to any Lorentz transformation. It is its only invariant.

Notice that the length, thus defined, of a four-vector may be either real, or purely imaginary, or nil, according as we have what was previously called a space-like, a time-like, or a singular vector.

If A, B be any pair of four-vectors, the sum of the products of their corresponding components is called their **scalar product**, and is denoted by (AB). Thus

$$(AB)=(A_xB_x+A_yB_y+A_zB_z+A_lB_l). \tag{45}$$

Guided by the analogy of ordinary vector-algebra, Sommerfeld defines then the **direction-cosine** of A relative to B, or *vice versa*, by writing

$$(AB)=|A|.|B|.\cos(A, B). \tag{45a}$$

Consequently, when

$$(AB)=0,$$

the four-vectors A, B are said to be **perpendicular** to one another This is identical with the previous definition of pairs of perpendicular vectors.

What Sommerfeld calls the 'vector product' of A, B cannot here occupy our attention. For such a product is a (special) six-vector, which as yet is unfamiliar to us.

As to the Lorentz transformation itself, it appears in Sommerfeld's treatment as a rotation of the system of four axes. Let P be any four-vector, and P_x, etc., its components along the old axes; then Sommerfeld defines the components of P along the new axes by

$$P_{x'}=P_x\cos(x', x)+P_y\cos(x', y)+P_z\cos(x', z)+P_l\cos(x', l), \tag{46}$$

and by similar formulae for $P_{y'}$, $P_{z'}$, $P_{l'}$. Here, the meanings of the cosines are defined by (45a). If the x'-axis is space-like, then the first three cosines in (46) are real, while $\cos(x', l)$ is purely imaginary, like P_l, so that $P_{x'}$ is real. Similarly, the y'- and z'-axes being space-like, $P_{y'}$, $P_{z'}$ will be real. And the l'-axis being time-

like, P_{r} will be purely imaginary. In order to show that the projection- or component-formulae (46), etc., are identical with those of the Lorentz transformation, Sommerfeld considers the particular case of rotation round the yz-plane, *i.e.* in the xl-plane, when

$$\cos(x', x) = \cos(l', l), \text{ say } = \cos\omega,$$

$$\cos(x', l) = -\cos(l', x) = \sqrt{1-\cos^2\omega} = \sin\omega,$$

$$\cos(y', y) = \cos(z', z) = 1,$$

while all other cosines vanish. Here we have, obviously, $\cos\omega > 1$, so that the angle ω, as well as its sine and tangent, are purely imaginary, and the absolute value of the latter is < 1. Consequently we can write

$$\tan\omega = \iota\beta, \quad \cos\omega = (1-\beta^2)^{-\frac{1}{2}} = \gamma, \quad \sin\omega = \iota\beta\gamma,$$

so that (46), etc., are at once reduced to the formulae (11), on p. 127, with the same meaning of ω, provided that the new system of space-axes $(x'y'z')$ moves relatively to the old one (xyz) with the uniform velocity $v = c\beta$ along the x-axis. There is in fact no difference whatever between Sommerfeld's and Minkowski's method of representing the relativistic transformation.

It is true that the systematic use of the four-dimensional vector language may offer some advantages, when compared with that based on the use of matrices. But, on the other hand, there are rather important arguments which may be brought forward in defence of the matrix-method. Thus, for instance, Sommerfeld's 'scalar product,' say (AB), is the same thing as Minkowski's product of the corresponding matrices, $W_1 \overline{W}_2$, (43). But whereas the invariance of $W_1 \overline{W}_2$ is seen at a glance, *viz.* by writing, in virtue of the fundamental formula (39),

$$W_1 A \,.\, \bar{A} \overline{W}_2 = W_1 \overline{W}_2,$$

the invariance of (AB) cannot be proved without splitting the four-vectors into their components and multiplying out expressions like (46) and adding them up. For Sommerfeld's only definition of 'scalar product' (45) is of such a character. It is essentially Cartesian, not vectorial. Of course, we know that, in three-dimensional space, the scalar product of a pair of vectors can be, and generally is, defined without any reference to axes, so that its invariance with respect to space rotations requires no proof. But

this does not by itself enable us to see the invariance of (AB), when we are asked to pass into the four-dimensional world, where our imagery fails us. Similar remarks could be made with respect to other points of Sommerfeld's method of treatment. But discussions of this kind need not detain us here any further.*

In the sequel we shall not avoid either of these two methods of analytic expression. In fact, we shall now and then profitably employ matrices as well as world-vectors. But principally we shall avail ourselves of the language of *Hamilton's quaternions*, the utility of which for relativistic purposes I have endeavoured to show in two papers.† I may notice that Minkowski himself (*Grundgleichungen*, p. 28, footnote) despised Hamilton's calculus of quaternions as 'too narrow and clumsy for the purpose' in question. But, notwithstanding that, I am still under the impression that quaternions are admirably suitable for most, if not for all, relativistic needs. We had a sample of the conciseness of Hamilton's language in Chapter II., when we saw how easily the four vector equations of the electron theory are condensed into a single quaternionic equation, *viz.* $D\mathbf{B} = C$. But in advocating here the cause of quaternions I am doing so not only because they furnish us very short formulae and simplify their handling. Quite independently of this, the quaternion seems to me intrinsically better adapted than the world-vector to express that 'union' of time and space which was (too strongly, perhaps) emphasized by Minkowski. For, although there is a certain union between the two, which manifests itself when we pass from one system to another, there is no total fusion. In each system, out of the four scalars x, y, z, l, the first three are more intimately bound to one another than any of them to the last one. The first three are artificial components of a vector, $\mathbf{r}$, which certainly is a more immediate entity than each of them. Now, in a four-vector, as well as in a matrix, x, y, z, l are, as it

* Nor can we enter here upon a paper of E. B. Wilson and G. N. Lewis, *Proc. Amer. Acad. of Arts and Sciences*, Vol. XLVIII., Nov. 1912, p. 389, in which an attempt is made to work out the four-dimensional vector-algebra and -analysis, *ab ovo*, starting from a number of quasi-geometric postulates.

† *Phil. Mag.*, Vol. XXIII., 1912, p. 790, and Vol. XXV., 1913, p. 135; also Bull. of the *Societas Scientiarum Varsaviensis*, Vol. IV. fasc. 9, communicated in November, 1911. I wish to mention here that Dr. G. F. C. Searle has drawn my attention to a paper of Prof. Conway, *Proc. Irish Acad.*, Vol. XXIX. Section A, March 1911, in which some of my results are arrived at. Particulars of comparison are left to the reader.

were, on entirely equal footing with one another, being the four 'components' of the former, or the four 'constituents' of the latter.* On the other hand, a quaternion q has a distinct vector part, $\mathrm{V}.q$ or simply $\mathrm{V}q$, and a scalar part, $\mathrm{S}q$, and none of the components of the former can be confounded with the latter. Now, the position of a particle is determined by a vector (in its ordinary sense), and its date by a scalar. What then more natural than to take the first as the V and to embody the second in the S of a quaternion? We could insist upon loosely juxtaposing both entities, and write simply

$$\mathbf{r}, \quad l.$$

But, if instead of the comma the plus sign is used, we have just enough of 'union' to express the relativistic standpoint, and yet enough distinction not to amalgamate time and space entirely.

Let us therefore combine the position vector $\mathbf{r}$ of a particle with its date, $l=\iota ct$, into a quaternion,

$$q=l+\mathbf{r}, \tag{47}$$

which, if it needed a name of its own, we might call the **position-quaternion.** Those who are particularly fascinated by the world-concept can consider this 'position' to be the 'position in the world.' But, in fact, the above provisional name is simply an abbreviation for 'position-date quaternion.'

The **conjugate** of q, *i.e.* Hamilton's $\mathrm{K}q$, will be denoted by q_c. Thus,

$$q_c=l-\mathbf{r}. \tag{47c}$$

The reader must not be afraid of quaternions. If he is familiar with the elements of ordinary vector-algebra, the following short remarks will enable him to understand thoroughly all of our subsequent calculations.

1. Without returning to Hamilton's original expression of a quaternion as the 'ratio' or the quotient of two vectors, he can conveniently define it from the outset as *the sum of a scalar and a vector*, using for the latter heavy type. Thus

$$a=\sigma+\mathbf{A}$$

will be a quaternion, whose *scalar part* is σ, and whose *vector part* is $\mathbf{A}$,

$$\mathrm{S}a=\sigma, \quad \mathrm{V}a=\mathbf{A}.$$

* It is true, that the fourth, l, is imaginary, while the first three are real, but this does not seem to me to emphasize the distinction sufficiently.

2. The *conjugate* a_c of the quaternion a is defined, as above, by

$$a_c = \sigma - \mathbf{A},$$

i.e. by $a_c = \mathrm{S}a - \mathrm{V}a$.

3. Two quaternions a, b are said to be *equal* if both their scalars and their vectors are equal to one another. Thus,

$$a = b$$

means the same thing as

$$\mathrm{V}a = \mathrm{V}b \quad \text{and} \quad \mathrm{S}a = \mathrm{S}b.$$

4. Quaternions are *added* to one another by adding separately their scalars and their vectors. Thus

$$c = a + b$$

means the same thing as

$$\mathrm{S}c = \mathrm{S}a + \mathrm{S}b, \quad \mathrm{V}c = \mathrm{V}a + \mathrm{V}b.$$

Now, since the addition of scalars and the addition of vectors are both commutative, the *commutative* property belongs also to the sum of quaternions,

$$b + a = a + b.$$

And for the same reason the *associative* law holds for the sum of any number of quaternions. Thus

$$a + [b + c] = [a + b] + c,$$

so that both sides may be simply written $a + b + c$.

5. *Subtraction* of quaternions, and the change of the sign of a quaternion are at once reduced to the same operations applied to scalars and vectors. Thus, if $a = \sigma + \mathbf{A}$,

$$-a = -\sigma - \mathbf{A}.$$

Also, by **4**, $a - b = -b + a$.

6. Two quaternions, $a = \sigma + \mathbf{A}$ and $b = \tau + \mathbf{B}$, are *multiplied* by the formula

$$ab = \sigma\tau + \tau\mathbf{A} + \sigma\mathbf{B} + \mathbf{AB},$$

where the first three terms require no further explanation, and the last is defined to be a quaternion

$$\mathbf{AB} = \mathrm{V}\mathbf{AB} + \mathrm{S}\mathbf{AB},$$

such that $\mathrm{V}\mathbf{AB}$ is identical with the 'vector product' and $\mathrm{S}\mathbf{AB}$ is the *negative* 'scalar product,' both supposed to be known from ordinary vector algebra. Thus, in our usual notation,

$$\mathbf{AB} = \mathrm{V}\mathbf{AB} - (\mathbf{AB}).$$

The *minus* sign is introduced to suit the whole of Hamilton's calculus; I do not think there is any trouble in doing so. Ultimately, the *product* ab of a pair of quaternions is given by

$$\mathrm{S}ab = \sigma\tau - (\mathbf{AB}),$$

$$\mathrm{V}ab = \tau\mathbf{A} + \sigma\mathbf{B} + \mathrm{V}\mathbf{AB}.$$

Thus, ab, and similarly, the product of any number of quaternions, is again a quaternion, with uniquely determinate vector and scalar parts.

Both $(\mathbf{AB})$ and $\mathrm{V}\mathbf{AB}$ being distributive, quaternion multiplication is *distributive*, *i.e.*

$$a[b+c] = ab + ac,$$

$$[b+c]a = ba + ca.$$

It can easily be shown that it is also *associative*, *i.e.* that

$$a \,.\, bc = ab \,.\, c,$$

so that both sides may be simply written abc. The same thing is true of the product of any number of quaternions. It is chiefly this associative property which makes Hamilton's calculus so powerful.

From the above formulae we see that

$$\mathrm{S}ba = \mathrm{S}ab,$$

because $(\mathbf{AB})$, like $\sigma\tau$, is commutative. On the other hand we have, generally,

$$\mathrm{V}ba \neq \mathrm{V}ab,$$

because $\mathrm{V}\mathbf{BA} = -\mathrm{V}\mathbf{AB}$.

Thus, multiplication of quaternions is, generally, **not commutative**,

$$ba \neq ab.$$

It becomes commutative only when $\mathrm{V}\mathbf{AB}$ vanishes, *i.e.* when $\mathbf{A} \parallel \mathbf{B}$, or $\mathrm{V}a \parallel \mathrm{V}b$. This is, for instance, the case for a pair of conjugate quaternions, and, consequently, we have, for any quaternion a,

$$aa_c = a_c a.$$

7. Writing again $a = \sigma + \mathbf{A}$, we have, by **6**,

$$aa_c = a_c a = \sigma^2 + \cancel{\mathbf{A}^2}\, A^2,$$

where $A^2 = (\mathbf{AA})$. Thus aa_c is always a pure scalar. Its square root is called the **tensor** of the quaternion a, and is denoted by $\mathrm{T}a$,

$$\mathrm{T}a = (\sigma^2 + A^2)^{\frac{1}{2}}.$$

If it is real, the positive value of the root is taken, and if purely imaginary, the positive imaginary value of the root is taken. (In cases of complex values, when ambiguity of T might arise, special explanations

will be given.) But the chief thing is to keep in mind the formula for the square of the tensor,

$$(\mathrm{T}a)^2 = aa_c = a_c a,$$

which is called the **norm** of the quaternion a.

Let $\mathbf{a}$ be the unit of $\mathbf{A}$, so that $(\mathbf{aa}) = 1$ and $\mathbf{A} = A\mathbf{a}$. (There is, I hope, no danger of confounding the quaternion a with the absolute value of $\mathbf{a}$, which is 1.) Then the quaternion a can be written

$$a = \mathrm{T}a \,.\, [\cos\alpha + \mathbf{a}\sin\alpha],$$

or, by an obvious analogy,

$$a = \mathrm{T}a \,.\, e^{\alpha\mathbf{a}},*$$

where e is the basis of natural logarithms. The factor of $\mathrm{T}a$, which is a quaternion of unit tensor or a **unit quaternion**, is called **the versor of the quaternion** a, and is denoted by $\mathrm{U}a$, so that $a = \mathrm{T}a \,.\, \mathrm{U}a$. The unit vector $\mathbf{a}$ is called the **axis** of the quaternion a, and the angle α, which can be real or imaginary, is called the **angle** of the quaternion a.

Thus, conjugate quaternions may be described as quaternions having equal tensors and equal angles, but opposite axes.

Notice that if (as in the case of our above q) σ is imaginary and $\mathbf{A}$ real, or *vice versa*, the tensor of a may vanish, though a is not simply 'zero,' but a definite quaternion having a certain axis and a certain angle. Such a quaternion was called by Hamilton a *nullifier*, and by Cayley a *nullitat.* In our physical applications we shall not avail ourselves of either of these names, but shall adopt for such quaternions the name **singular**, already used for the corresponding world-vectors.

8. The following rule, which will be often required, can easily be proved:

The conjugate of the product of any number of quaternions is the product of their conjugates in the **reversed** *order.*

Thus, if

$$m = ab,$$

then

$$m_c = b_c a_c.$$

9. Finally, as regards *division* by quaternions, it may be entirely reduced to multiplication by what are called their reciprocals.

The **reciprocal** of a quaternion a is again a certain quaternion, which is denoted by a^{-1} and which is defined by the equation

$$a^{-1}a = 1.$$

* But this analogy cannot be pushed so far as to write in the expression of a product of two quaternions a, b

$$e^{\alpha\mathbf{a}+\beta\mathbf{b}}$$

and to invert the order of addends in the exponent. For, unless $\mathbf{a} \parallel \mathbf{b}$, the product ab is *not* commutative.

Multiply both sides by a_c as a postfactor. Then $a^{-1}aa_c = a_c$, and, by **7**,

$$a^{-1} = \frac{a_c}{(\mathrm{T}a)^2} = \frac{1}{\mathrm{T}a} \cdot \mathrm{U}a_c = \frac{1}{\mathrm{T}a} \cdot e^{-\alpha\mathbf{a}}.$$

Thus, the reciprocal of a quaternion is its conjugate divided by its norm. In other words, the reciprocal of a has the reciprocal tensor, the opposite axis and the same angle as a.

Consequently, we can also write

$$aa^{-1} = 1.$$

Thus, if we have an equation

$$am = b$$

and we wish to isolate m, we have only to multiply both sides by a^{-1} as prefactor, obtaining

$$m = a^{-1}b.$$

Similarly, if

$$na = b,$$

we shall have

$$n = ba^{-1}.$$

Notice, in particular, that the reciprocal of a *unit quaternion* is at the same time its *conjugate*.

10. The differentiation of quaternions, with respect to time or position in space, does not require any explanations. The definition of 'curl' and 'div' being supposed known from usual vector-analysis, it will be enough to remember here what was said already in Chap. II., namely that ∇, the vector part of our D, when applied to a vector $\mathbf{A}$, gives

$$\nabla\mathbf{A} = \mathrm{V}\nabla\mathbf{A} + \mathrm{S}\nabla\mathbf{A} = \mathrm{V}\nabla\mathbf{A} - (\nabla\mathbf{A})$$

(as in **4**, because ∇ apart from its differentiating properties is to be treated as an ordinary vector), or ultimately

$$\nabla\mathbf{A} = \mathrm{curl}\,\mathbf{A} - \mathrm{div}\,\mathbf{A}.$$

For all of our purposes we shall hardly want more than is given in the above ten sections,—which will in the sequel be shortly referred to as 'Quat. **1**, **2**, etc.'

Returning to our position-quaternion q, let us write its S'-correspondent, or the transformed quaternion

$$q' = l' + \mathbf{r}'. \tag{47'}$$

Since $l = \iota ct$, *i.e.* $t = -\iota l/c$, we have, by (1*b*), p. 124, and denoting now the unit of $\mathbf{v}$ by $\mathbf{u}$,

$$\left.\begin{aligned} l' &= \gamma[l - \iota\beta(\mathbf{ur})] \\ \mathbf{r}' &= \epsilon\mathbf{r} + \iota\beta\gamma l\mathbf{u}. \end{aligned}\right\} \tag{48}$$

Here, it will be remembered, ϵ is the longitudinal stretcher, whose developed form is, by (2),

$$\epsilon = 1 + (\gamma - 1)\mathbf{u}(\mathbf{u} \quad .$$

Now, such being the scalar and the vector parts of q' in terms of those of q, we can easily find a quaternion Q such that

$$q' = QqQ. * \tag{49}$$

First of all, since we know that $l^2 + r^2$ is an invariant, or that $\mathrm{T}q' = \mathrm{T}q$, we can at once take for Q a *unit* quaternion,

$$Q = \cos\theta + \mathbf{a} \, . \sin\theta.$$

Thus, we have only to find the angle and the axis of Q in terms of β and $\mathbf{u}$. Now, developing the triple product in (49), we obtain easily, by Quat. 6,

$$l' \equiv SQqQ = \cos(2\theta) \, . \, l - \sin(2\theta) \, . \, (\mathbf{ar}),$$

$$\mathbf{r}' \equiv \mathrm{V}QqQ = \mathbf{r} - 2\sin^2\theta \, . \, \mathbf{a}(\mathbf{ar}) + \sin(2\theta) \, . \, l\mathbf{a},$$

whence, comparing with (48),

$$\mathbf{a} = \mathbf{u}; \quad \cos(2\theta) = \gamma; \quad \sin(2\theta) = \iota\beta\gamma,$$

and $1 - 2\sin^2\theta \, . \, \mathbf{a}(\mathbf{a} = \epsilon$, *i.e.* $2\sin^2\theta = 1 - \gamma$, which is identical with the third of the above equations, and this, again, says the same thing as the second. Thus, all conditions are satisfied at once, and we have ultimately

$$\mathbf{a} = \mathbf{u} \quad \text{and} \quad \theta = \tfrac{1}{2}\arctan(\iota\beta) = \tfrac{1}{2}\omega,$$

where ω is the (imaginary) angle of rotation, as previously defined. [Cf. (10), p. 127.] To resume:

The position-quaternion q is transformed by the operator

$$Q[\quad]Q,$$

the vacant place being destined for the operand.

The axis of the unit quaternion Q is $\mathbf{u}$, *the unit of* $\mathbf{v}$, *and its angle is half that of Minkowski's imaginary angle of rotation, i.e.*

$$Q = \cos\frac{\omega}{2} + \mathbf{u} \, . \sin\frac{\omega}{2} = e^{\frac{\omega}{2}\mathbf{u}}. \tag{50}$$

* As regards the reason why particularly *this* form, involving a quaternionic prefactor and postfactor, is sought for, see my paper in *Phil. Mag.*, Vol. XXIII., quoted before, where I gave references going back to Cayley's original discovery (1854).

Another form of this quaternion is

$$Q=\left(\frac{1+\gamma}{2}\right)^{\frac{1}{2}}+\mathbf{u}\,.\left(\frac{1-\gamma}{2}\right)^{\frac{1}{2}}. \tag{50a}$$

Observe that, γ being >1, the vector part of Q is imaginary, while its scalar part is real.

Since Q is a unit quaternion, we have $Q^{-1}=Q_c$, or

$$QQ_c=Q_cQ=1,$$

a property which we shall constantly use. Thus, to obtain from (49) the inverse transformation, multiply both sides by Q_c as a post- and a prefactor. Then the result will be

$$q=Q_cq'Q_c, \tag{49a}$$

as it should be, since Q_c is obtained from Q by a reversal of $\mathbf{u}$ or $\mathbf{v}$. Again, to see once more, or to verify, the invariance of

$$(\mathrm{T}q)^2=qq_c=r^2+l^2=r^2-c^2t^2, \tag{51}$$

take the conjugate of (49), which, by Quat. 8, is

$$q'_c=Q_cq_cQ_c.$$

Now, by the same formula (49), and by the associative law,

$$q'q'_c=QqQQ_cq_cQ_c=Qqq_cQ_c.$$

But, since qq_c is a scalar, it may be written before the Q, or if you wish, after the Q_c, so that

$$q'q'_c=qq_cQQ_c=qq_c,$$

Q.E.D.

We shall see later on, when we come to consider products of two, or more, of such quaternions, that they are transformed with equal ease.

Consecutive transformations assume the following simple form. Let $\mathbf{v}_1=v_1\mathbf{u}_1$ be the velocity of S' relative to S, and $\mathbf{v}_2=v_2\mathbf{u}_2$ the velocity of S'' relative to S', and let Q_1, Q_2 be the corresponding transforming quaternions, *i.e.*

$$Q_1=e^{\frac{\omega_1}{2}\mathbf{u}_1},\quad Q_2=e^{\frac{\omega_2}{2}\mathbf{u}_2}.$$

Then

$$q' = Q_1 q Q_1$$

and

$$q'' = Q_2 q' Q_2 = Q_2 Q_1 q Q_1 Q_2,$$

so that the compound transformer is

$$Q_2 Q_1 [\quad] Q_1 Q_2.$$

In general, for non-parallel axes $\mathbf{u}_1$, $\mathbf{u}_2$,

$$Q_2 Q_1 \neq Q_1 Q_2,$$

so that the compound transformer has not the form $Q[\quad]Q$. This is but the quaternionic expression of the fact, to be considered fully in the following chapter, that a pair of consecutive three-parametric Lorentz transformations, (48), does not generally give again such a transformation, but is equivalent to (48) combined with a pure rotation in ordinary three-dimensional space. In other words, the transformations (48) do not constitute a group. But, as we saw before, they contain sub-groups, namely for parallel velocities. Then, and only then, $Q_2 Q_1$ becomes equal to $Q_1 Q_2$, and the compound transformer assumes the form $Q[\quad]Q$. Suppose, for instance, that the velocities $\mathbf{v}_1$, $\mathbf{v}_2$, being parallel, are also concurrent with one another, *i.e.* that

$$\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}.$$

Then

$$Q = e^{\frac{\omega}{2}\mathbf{u}} = e^{\frac{1}{2}(\omega_1+\omega_2)\mathbf{u}},$$

so that the previous formula for the composition of parallel velocities, $\omega = \omega_1 + \omega_2$, follows from the quaternionic form immediately.

Imitating the name 'world-vector,' we could now call q, or q_c, the standard of world-quaternions. But the more modest name **physical quaternion** will do as well. Also, to begin with, no further specification of the 'kind' is needed. But it may be convenient to have a pair of short symbols, in order to compare any quaternions with respect to their relativistic behaviour. By writing

$$X \sim q,$$

we shall understand that the quaternion X is covariant or, **equally transformed**, with q, *i.e.* that

$$X' = QXQ$$

without taking into account the structure of X. And if X has also the **structure** of q, that is to say, if it has a purely imaginary scalar and a real vector,* then we shall write

$$X \simeq q.$$

The latter will then be equivalent to saying that X is a physical quaternion, viz. covariant with q. This being the case, the conjugate of X will, of course, be also a physical quaternion, *e.g.*

$$X_c \simeq q_c.$$

The same notation we shall extend to quaternionic operators. Thus, as we saw, $\partial/\partial l$ and ∇, the scalar and the vector parts of the operator D, are transformed like l, $\mathbf{r}$, the scalar and the vector parts of the position-quaternion, *i.e.*

$$D' = QDQ, \tag{52}$$

and similarly, $D_c = \partial/\partial l - \nabla$ being the conjugate operator,

$$D_c' = Q_c D_c Q_c. \tag{52c}$$

But D has also the same structure as q. Consequently, apart from its differentiating properties, D behaves as a genuine physical quaternion, or

$$D \simeq q.$$

Analogously to Minkowski's classification of four-vectors, we may call any physical quaternion X a **space-like**, or a **time-like**, or finally a **singular quaternion**, according as its norm, $(\mathrm{T}X)^2 = XX_c$, is positive, or negative, or zero.

But it does not seem desirable to dwell any longer upon the formal side of the subject until our stock of materials has been somewhat enlarged. For as yet we have only one physical quaternion, namely q.

* If the reverse is the case, then ιX will have the structure of q.

NOTE TO CHAPTER V.

(To page 143.) A **matrix** is any rectangular array of magnitudes or, more generally, of symbols either of magnitude or of operation, each of which has its assigned place, *i.e.* belongs to a given row and a given column. Thus

$$A=\left|\begin{array}{llll} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}, & a_{m2}, & \ldots & a_{mn} \end{array}\right|$$

is a matrix of m *rows* and n *columns*. The first index of any constituent $a_{\iota\kappa}$ denotes the row, and the second the column to which it belongs.

The matrix whose rows are the columns of A is called the **transposed** of A, and is denoted by $\bar{A}$. Thus, A being as above,

$$\bar{A}=\left|\begin{array}{llll} a_{11}, & a_{21}, & \ldots & a_{m1} \\ a_{12}, & a_{22}, & \ldots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n}, & a_{2n}, & \ldots & a_{mn} \end{array}\right|.$$

To specify the number of rows and columns of a matrix we may conveniently attach to its symbol a pair of indices. Thus, A will be A_{mn}, and similarly $\bar{A}=\bar{A}_{nm}$. Or we may say, equivalently, that A is a matrix of $m \times n$ constituents, and $\bar{A}$ a matrix of $n \times m$ constituents.

If we have a pair of matrices $A=A_{mn}$ and $B=B_{mn}$, then the matrix $C=C_{mn}$, whose constituents are sums of the corresponding constituents of A, B (*i.e.* $c_{\iota\kappa}=a_{\iota\kappa}+b_{\iota\kappa}$), is written

$$C=A+B.$$

If, in particular, $B=A$, the result of addition is written $2A$, and so on. Generally, if α be any number (or symbol of operation) and A any matrix, then the matrix C, whose constituents are $c_{\iota\kappa}=\alpha a_{\iota\kappa}$, is called the product of α into A, and is denoted by αA.

If the matrix B *has as many rows as* A *has columns*, *i.e.* if

$$A=A_{mn}, \quad B=B_{np}$$

(where p may be equal to or different from m), then the matrix C, of which any constituent $c_{\iota\kappa}$ is equal to the sum of the products of the constituents of the ιth row of A into those of the κth column of B, is called the *product of* A *into* B, and is written

$$C=AB.$$

Thus, if A is as above, and if

$$B=\begin{vmatrix} b_{11}, & b_{12}, & \dots b_{1p} \\ b_{21}, & b_{22}, & \dots b_{2p} \\ \vdots & \vdots & \vdots \\ b_{n1}, & b_{n2}, & \dots b_{np} \end{vmatrix},$$

then

$$C=C_{mp}=A_{mn}B_{np}=\begin{vmatrix} c_{11}, & c_{12}, & \dots c_{1p} \\ c_{21}, & c_{22}, & \dots c_{2p} \\ \vdots & \vdots & \vdots \\ c_{m1}, & c_{m2}, & \dots c_{mp} \end{vmatrix},$$

where

$$c_{11}=a_{11}b_{11}+a_{12}b_{21}+\dots+a_{1n}b_{n1},$$

$$c_{12}=a_{11}b_{12}+a_{12}b_{22}+\dots+a_{1n}b_{n2},$$

and so on, generally

$$c_{\iota\kappa}=a_{\iota 1}b_{1\kappa}+a_{\iota 2}b_{2\kappa}+\dots+a_{\iota n}b_{n\kappa}.$$

Notice that, if $p \neq m$, the expression BA would be meaningless. But, since $\bar{B}=\bar{B}_{pn}$ and $\bar{A}=\bar{A}_{nm}$, we can have the product $\bar{B}\bar{A}$, which will be a matrix of $p \times m$ constituents. This, as can easily be seen, will be the transposed of AB, *i.e.*

$$\overline{AB}=\bar{B}\bar{A}.$$

Compare this property with Quat. 8. p. 154.

Since $AB=C_{mp}$, it can be multiplied into a third matrix $D=D_{pq}$, thus giving rise to $AB \,.\, D$, which will be a matrix of $m \times q$ constituents. It can be proved that for such products *the associative law holds* (supposing, of course, that the constituents themselves, which generally can be operators, obey this law), *i.e.*

$$AB \,.\, D=A \,.\, BD.$$

Hence, both sides may be simply written ABD. The same property belongs to the product of any number of matrices. Thus

$$A_{mn}B_{np}D_{pq}\dots M_{xy}N_{yz}=R_{mz}$$

will be a definite matrix of $m \times z$ constituents, independent of the grouping of the factors. Notice the analogy with quaternionic products.

Let each of the constituents of *the principal diagonal* (from left uppermost to right lowermost) of a square matrix $U=U_{nn}$ be equal 1, *i.e.*

$$u_{11}=u_{22}=\dots=u_{nn}=1,$$

and let each of its remaining constituents $u^{\iota\kappa}$ be *zero*. Then, if M be any matrix of n rows,

$$UM=M.$$

In view of this property, U is called a **unit-matrix**, and may be simply denoted by 1.

Now, let M be any *square* matrix. Then the determinant formed of its constituents is called the determinant of M, and is shortly written $\det M$. Suppose that $\det M$ does not vanish. Then there exists a definite matrix which, multiplied into M, gives a unit matrix or simply 'unity.' This matrix is called the **reciprocal** of M, and is denoted by M^{-1}. The above definition is written shortly

$$M^{-1}M = 1,$$

where 1 stands for U_{nn}. The reciprocal is, of course, as M itself, a square matrix of $n \times n$ constituents.

Other particulars concerning matrices will be given incidentally, as the need arises in the subject under consideration.

CHAPTER VI.

COMPOSITION OF VELOCITIES AND THE LORENTZ GROUP.

Consider a particle moving about in an arbitrary manner in the system S', which in its turn moves with uniform velocity $\mathbf{v}$ relatively to the system S. Let $\mathbf{p}'$ be the instantaneous velocity of the particle from the point of view of the S'-observers, *i.e.* let at the instant t'

$$\frac{d\mathbf{r}'}{dt'}=\mathbf{p}'.$$

What is the velocity $\mathbf{p}$ of this particle from the S-standpoint, at the instant t corresponding to t'?

To answer this simple but very fundamental question of relativistic kinematics, use the form (1b), Chap. V., of the Lorentz transformation. Then its inverse will be, as in (1b'),

$$\mathbf{r}=\epsilon\mathbf{r}'+\gamma\mathbf{v}t',$$

$$t=\gamma\left[t'+\frac{1}{c^2}(\mathbf{v}\mathbf{r}')\right],$$

and, since $d\epsilon\mathbf{r}'=\epsilon d\mathbf{r}'$ and $d(\mathbf{v}\mathbf{r}')=(\mathbf{v}\,d\mathbf{r}')$,

$$\mathbf{p}=\frac{d\mathbf{r}}{dt}=\frac{\epsilon\,d\mathbf{r}'+\gamma\mathbf{v}\,dt'}{\gamma\left[dt'+\frac{1}{c^2}(\mathbf{v}\,d\mathbf{r}')\right]}.$$

Divide the numerator and the denominator on the right by dt', and remember the meaning of $\mathbf{p}'$. Then the required velocity will follow at once under the simple form

$$\mathbf{p}=\frac{\gamma\mathbf{v}+\epsilon\mathbf{p}'}{\gamma\left[1+\frac{1}{c^2}(\mathbf{v}\mathbf{p}')\right]}. \qquad (1a)$$

This is the vectorial expression of Einstein's famous *Addition Theorem.**

As before, $\gamma=(1-\beta^2)^{-\frac{1}{2}}$, $\beta=v/c$, and ϵ is the longitudinal stretcher of ratio γ. Thus, in Cartesians, with x measured along $\mathbf{v}$, (1*a*) will become

$$p_x=\frac{v+p_x'}{1+vp_x'/c^2},\quad p_y=\frac{p_y'}{\gamma(1+vp_x'/c^2)},\quad p_z=\frac{p_z'}{\gamma(1+vp_x'/c^2)}.$$

But having explained this for the non-vectorial reader, we shall henceforth use the above short formula.

By writing $\mathbf{p}'$, $\mathbf{p}$ we wished to emphasize that the latter is the S-correspondent of the former. But we may as well look at $\mathbf{p}$ as the *resultant* of $\mathbf{v}$ and $\mathbf{p}'$, keeping in mind that the first of these component velocities is taken relatively to one system, S, and the second relatively to another† system S'. Then it may be more convenient to write for the velocities to be compounded $\mathbf{v}_1$, $\mathbf{v}_2$ (instead of $\mathbf{v}$, $\mathbf{p}'$), and for the resultant velocity $\mathbf{v}$ (instead of $\mathbf{p}$). Thus, attaching the correspondent index to γ and ϵ, we shall write

$$\mathbf{v}=\frac{\gamma_1\mathbf{v}_1+\epsilon_1\mathbf{v}_2}{\gamma_1\left[1+\frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right]}. \tag{1}$$

Notice that the resultant is, in general, a non-symmetrical function of the two component velocities. It is important to know which of these comes first, and which next. In Newtonian or classical kinematics the resultant is simply $\mathbf{v}_1$ *plus* $\mathbf{v}_2$ and at the same time $\mathbf{v}_2$ *plus* $\mathbf{v}_1$. Here the case is different. We may still speak of 'addition,' as a non-pedantic synonym of composition of velocities, but to avoid confusion we should employ instead of the ordinary + another symbol, say $\#$, and write the above $\mathbf{v}$, as given by (1),

$$\mathbf{v}_1 \# \mathbf{v}_2.$$

* 'Additionstheorem der Geschwindigkeiten,' *Ann. d. Phys.*, Vol. XVII., 1905; § 5.

† If both were taken with respect to the *same* system, then their resultant would, of course, be simply equal to their vector sum. But this is hardly worth mentioning. For all cases of composition of velocities, which have any *physical* interest, are of the type considered above, *viz.* imply component velocities referred to a chain of different systems: An object B moves in a given way relatively to A, a third object C moves relatively to B, and so on; find the motion of the last relative to the first.

Then the resultant of $\mathbf{v}_2$ and $\mathbf{v}_1$ (*i.e.* the *S*-velocity of a particle moving with velocity $\mathbf{v}_1$ relative to S', which in its turn moves with velocity $\mathbf{v}_2$ relative to S) would be

$$\mathbf{v}_2 \mathbin{+\!\!\!+} \mathbf{v}_1 = \frac{\gamma_2\mathbf{v}_2 + \epsilon_2\mathbf{v}_1}{\gamma_2\left[1 + \frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right]}, \tag{2}$$

where ϵ_2 is a stretcher acting along $\mathbf{v}_2$, of ratio γ_2.

In short, the relativistic composition of velocities is, generally speaking, **non-commutative.**

But it is interesting and, in view of what has to come later, useful to notice that the two vectors (1), (2), though differing in direction, are identical in their absolute magnitude. To see this, we have only to prove that the squares of the two vectors

$$\mathbf{a} = \mathbf{v}_1 + \frac{1}{\gamma_1}\epsilon_1\mathbf{v}_2$$

and

$$\mathbf{b} = \mathbf{v}_2 + \frac{1}{\gamma_2}\epsilon_2\mathbf{v}_1$$

are equal to one another. Now, by the elementary rules of vector algebra,

$$a^2 = v_1{}^2 + \frac{2}{\gamma_1}(\mathbf{v}_1 \cdot \epsilon_1\mathbf{v}_2) + \frac{1}{\gamma_1{}^2}(\epsilon_1\mathbf{v}_2)^2,$$

and, since ϵ_1 is a symmetrical vector-operator,

$$(\mathbf{v}_1 \cdot \epsilon_1\mathbf{v}_2) = (\epsilon_1\mathbf{v}_1 \cdot \mathbf{v}_2) = \gamma_1(\mathbf{v}_1\mathbf{v}_2).$$

Again, denoting by θ the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\left(\frac{1}{\gamma_1}\epsilon_1\mathbf{v}_2\right)^2 = v_2{}^2\left[\cos^2\theta + \frac{1}{\gamma_1{}^2}\sin^2\theta\right] = v_2{}^2\left[1 - \beta_1{}^2\sin^2\theta\right].$$

Hence

$$a^2 = v_1{}^2 + v_2{}^2 + 2(\mathbf{v}_1\mathbf{v}_2) - \frac{1}{c^2}v_1{}^2v_2{}^2\sin^2\theta = (\mathbf{v}_1 + \mathbf{v}_2)^2 - \frac{1}{c^2}(\mathrm{V}\mathbf{v}_1\mathbf{v}_2)^2,$$

and this, being a symmetrical function of $\mathbf{v}_1$, $\mathbf{v}_2$, is at the same time the value of b^2. Q.E.D.

Thus we have the general property of relativistic composition of velocities:

$$(\mathbf{v}_1 \mathbin{+\!\!\!+} \mathbf{v}_2)^2 = (\mathbf{v}_2 \mathbin{+\!\!\!+} \mathbf{v}_1)^2. \tag{3}$$

The common value of these scalars is, by (1) and by the formula just found for a^2,

$$v^2 = \frac{(\mathbf{v}_1+\mathbf{v}_2)^2 - \frac{1}{c^2}(V\mathbf{v}_1\mathbf{v}_2)^2}{\left[1+\frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right]^2}. \tag{4}$$

This is Einstein's famous formula for the square of the resultant velocity, written vectorially.

Before passing to give a few examples and a certain very remarkable geometric representation of the Addition Theorem (1), let us approach the question of composition of velocities from another side, *e.g.* by considering a pair of consecutive Lorentz transformations.

Let again $\mathbf{v}_1$ be the velocity of S' relative to S, but instead of our particle take a third system S'' moving relatively to S' with the velocity $\mathbf{v}_2$, the former velocity being taken from the S-standpoint and the latter from the S'-point of view, both being now uniform translational velocities. Let γ_1, ϵ_1 and γ_2, ϵ_2 be the corresponding meanings of γ, ϵ. Then, t', $\mathbf{r}'$ being the time and the space-vector in S', and t'', $\mathbf{r}''$ the time and the space-vector in S'', we shall have, by (1*b*), Chap. V.,

$$\mathbf{r}' = \epsilon_1\mathbf{r} - \mathbf{v}_1\gamma_1 t; \quad t' = \gamma_1\left[t - \frac{1}{c^2}(\mathbf{v}_1\mathbf{r})\right] \tag{5_1}$$

and

$$\mathbf{r}'' = \epsilon_2\mathbf{r}' - \mathbf{v}_2\gamma_2 t'; \quad t'' = \gamma_2\left[t' - \frac{1}{c^2}(\mathbf{v}_2\mathbf{r}')\right]. \tag{5_2}$$

Introduce the values (5_1) of $\mathbf{r}'$ and t' into (5_2), and remember that, ϵ_1 being a symmetrical vector operator, $(\mathbf{v}_2 \,.\, \epsilon_1\mathbf{r}) = (\epsilon_1\mathbf{v}_2 \,.\, \mathbf{r})$. Then the result will be

$$\left.\begin{aligned} \mathbf{r}'' &= \epsilon_2\epsilon_1\mathbf{r} + \frac{1}{c^2}\gamma_1\gamma_2\mathbf{v}_2(\mathbf{v}_1\mathbf{r}) - \gamma_1\left[\epsilon_2\mathbf{v}_1 + \gamma_2\mathbf{v}_2\right]t \\ t'' &= \gamma_1\gamma_2\left[1 + \frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right]t - \frac{1}{c^2}\gamma_2\left[(\epsilon_1\mathbf{v}_2 \,.\, \mathbf{r}) + \gamma_1(\mathbf{v}_1\mathbf{r})\right]. \end{aligned}\right\} \tag{6}$$

The Lorentz transformations hitherto considered, of which (5_1) and (5_2) are individual cases, involve three scalar parameters (v_x, v_y, v_z) or one vectorial parameter $\mathbf{v}$. Let us therefore denote any one of these transformations by $L(\mathbf{v})$. Thus, the two above component transformations will be $L(\mathbf{v}_1)$, $L(\mathbf{v}_2)$, and their resultant,

i.e. the first followed by the second, or the transformation (6), may be written $L(\mathbf{v}_2)L(\mathbf{v}_1)$.

We know that any $L(\mathbf{v})$ leaves invariant the quadratic expression

$$x^2+y^2+z^2+l^2,$$

and can therefore be considered as a rotation in the four-dimensional world. But it is not the most general rotation, since it does not include the rotation round the time-axis, *i.e.* a rotation of the space-framework, or an equivalent rotation of the three-dimensional vectors. If any transformation $L(\mathbf{v})$ is followed by such a rotation of $\mathbf{r}'$, which does not change the value of $r'^2=x'^2+y'^2+z'^2$, then the above quadratic expression will, obviously, continue to be an invariant. Let Ω be a purely rotating operator, or what Gibbs* called a 'versor,' *i.e.* such a linear vector operator that, for any vector $\mathbf{R}$,

$$(\Omega\mathbf{R})^2=R^2.$$

Then the amplified or, as it is sometimes called, **the general Lorentz transformation** will be given by

$$\left.\begin{aligned}\mathbf{r}'&=\Omega\left[\epsilon\mathbf{r}-\mathbf{v}\gamma t\right],\\ t'&=\gamma\left[t-\frac{1}{c^2}(\mathbf{vr})\right].\end{aligned}\right\}\qquad L(\mathbf{v},\,\Omega)$$

Since Ω involves three scalar data, viz. one for its angle and two for its axis, $L(\mathbf{v},\,\Omega)$ will be a *six-parametric* transformation. Thus, the above symbol $L(\mathbf{v})$ of the *special* Lorentz transformation stands for $L(\mathbf{v},\,1)$. Notice that the scalar product of two vectors, *e.g.* $(\mathbf{vr})$, is not changed at all by a pure space-rotation. This is the reason that Ω does not enter into the expression for t', and would not enter into it even if the rotation preceded the special Lorentz transformation.

Let us now return to our $L(\mathbf{v}_2)L(\mathbf{v}_1)$, as given by the formulae (6).

We have seen in the last chapter that, if the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are *parallel* to one another, the resultant transformation is again a special Lorentz transformation, *i.e.*

$$L(\mathbf{v}_2)L(\mathbf{v}_1)=L(\mathbf{v}),$$

where $\mathbf{v}\parallel\mathbf{v}_1\parallel\mathbf{v}_2$. Now, it can easily be shown that this is the case **only** for $\mathbf{v}_1\parallel\mathbf{v}_2$.

*J. Willard Gibbs, *Scientific Papers*, Vol. II. p. 64.

In fact, suppose that (6) is an $L(\mathbf{v})$, that is to say, suppose that there is a vector $\mathbf{v}$ (with the corresponding γ and ϵ), such that

$$\mathbf{r}'' = \epsilon\mathbf{r} - \mathbf{v}\gamma t; \quad t'' = \gamma\left[t - \frac{1}{c^2}(\mathbf{vr})\right].$$

Then, remembering that this has to coincide with (6) for every $\mathbf{r}$ (as well as for every t) and taking, for instance, $r = 0$, you will obtain, from the first of (6),

$$\gamma\mathbf{v} = \gamma_1[\epsilon_2\mathbf{v}_1 + \gamma_2\mathbf{v}_2],$$

and at the same time, from the second of (6),

$$\gamma\mathbf{v} = \gamma_2[\epsilon_1\mathbf{v}_2 + \gamma_1\mathbf{v}_1],$$

and, consequently,

$$\gamma_1[\epsilon_2\mathbf{v}_1 + \gamma_2\mathbf{v}_2] = \gamma_2[\epsilon_1\mathbf{v}_2 + \gamma_1\mathbf{v}_1].$$

Now, this equation cannot be satisfied unless $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. To see this, call $\mathbf{l}_1$ and $\mathbf{n}_1$ the parts of $\mathbf{v}_1$ taken along and normal to $\mathbf{v}_2$, and similarly $\mathbf{l}_2$ and $\mathbf{n}_2$ the parts of $\mathbf{v}_2$ taken along and normal to $\mathbf{v}_1$, and write $\mathbf{v}_1 = \mathbf{l}_1 + \mathbf{n}_1$, $\mathbf{v}_2 = \mathbf{l}_2 + \mathbf{n}_2$. Then, remembering that ϵ_1, ϵ_2 are longitudinal stretchers, the above equation will assume the form

$$\gamma_1[\gamma_2\mathbf{l}_1 + \mathbf{n}_1 + \gamma_2\mathbf{l}_2 + \gamma_2\mathbf{n}_2] = \gamma_2[\gamma_1\mathbf{l}_2 + \mathbf{n}_2 + \gamma_1\mathbf{l}_1 + \gamma_1\mathbf{n}_1]$$

or

$$\gamma_1(1 - \gamma_2)\mathbf{n}_1 = \gamma_2(1 - \gamma_1)\mathbf{n}_2.$$

Hence, either $\gamma_1 = \gamma_2 = 1$, which corresponds to the trivial case $v_1 = v_2 = 0$, or $\mathbf{n}_1 \parallel \mathbf{n}_2$, and consequently also $\mathbf{v}_1 \parallel \mathbf{v}_2$. Q.E.D.

Thus, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel to one another, the resultant transformation (6) is *not* an $L(\mathbf{v})$. In other words, the class of ∞^3 transformations $L(\mathbf{v})$ does not constitute a group, although it contains one-parametric subgroups, each ranging over parallel velocities.

But the six-parametric transformations $L(\mathbf{v}, \Omega)$ do constitute a group, *i.e.*

$$L(\mathbf{v}_2, \Omega_2)L(\mathbf{v}_1, \Omega_1) = L(\mathbf{v}, \Omega),$$

for any pair of velocities and any pair of versors, and hence, in particular, also for $\Omega_1 = 1$, $\Omega_2 = 1$, as in our case. For non-parallel velocities, then, our $L(\mathbf{v}_2)L(\mathbf{v}_1)$ is not again an $L(\mathbf{v})$, but it is an

$L(\mathbf{v}, \Omega)$ with a certain space-rotation,* to be determined. In fact, the formulae (6) are of the form

$$\mathbf{r}'' = \Omega[\epsilon\mathbf{r} - \mathbf{v}\gamma t] = \Omega\epsilon r - \gamma t\Omega\mathbf{v}$$

$$t'' = \gamma\left[t - \frac{1}{c^2}(\mathbf{vr})\right],$$

where $\Omega \neq 1$.

A comparison with (6) will give us the four equations

$$\Omega\epsilon = \epsilon_2\epsilon_1 + \frac{\gamma_1\gamma_2}{c^2}\mathbf{v}_2(\mathbf{v}_1 \qquad (a)$$

$$\gamma = \gamma_1\gamma_2\left[1 + \frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right] \qquad (b)$$

$$\Omega\mathbf{v} = \frac{\gamma_1}{\gamma}[\epsilon_2\mathbf{v}_1 + \gamma_2\mathbf{v}_2] \qquad (c)$$

$$\mathbf{v} = \frac{\gamma_2}{\gamma}[\epsilon_1\mathbf{v}_2 + \gamma_1\mathbf{v}_1]. \qquad (d)$$

From (*b*), (*d*) we have at once the resultant velocity, of S'' relative to S,

$$\mathbf{v} = \mathbf{v}_1 \# \mathbf{v}_2 = \frac{\gamma_1\mathbf{v}_1 + \epsilon_1\mathbf{v}_2}{\gamma_1\left[1 + \frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right]},$$

identical with (1), which was obtained by differentiation. The verification that γ, as given by (*b*), is equal to $(1 - v^2/c^2)^{-\frac{1}{2}}$, is left to the reader. Again, the right side of (*c*) is what $\mathbf{v}$ becomes by permutation of $_{1,\,2,}$ so that

$$\Omega\mathbf{v} = \Omega[\mathbf{v}_1 \# \mathbf{v}_2] = \mathbf{v}_2 \# \mathbf{v}_1, \qquad (7)$$

and this agrees with the nature of the operator Ω. For, as was shown explicitly, the tensors of the two resultant velocities are equal; cf. (3). Thus, Ω *turns* $\mathbf{v}_1 \# \mathbf{v}_2$ *into* $\mathbf{v}_2 \# \mathbf{v}_1$. The equation (7),

* In four-dimensional language the case under consideration may be expressed as follows. Call $\mathbf{t}$ the time-axis in Minkowski's world. Then $L(\mathbf{v}_1)$ will be a rotation in the plane $\mathbf{t}, \mathbf{v}_1$; similarly, $L(\mathbf{v}_2)$ will be a rotation in the plane $\mathbf{t}, \mathbf{v}_2$. Now, if $\mathbf{v}_2 \parallel \mathbf{v}_1$, the resultant transformation $L(\mathbf{v}_2)L(\mathbf{v}_1)$ will again be a rotation in the plane $\mathbf{t}, \mathbf{v}_1$. But if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, the resultant four-dimensional rotation will also have a component 'round $\mathbf{t}$,' *i.e.* $L(\mathbf{v}_2)L(\mathbf{v}_1)$ will involve also a pure (three-dimensional) space-rotation.

of course, does not by itself suffice for a complete determination of the operator, for it states the result of its application to a special vector $\mathbf{v}$ only. But we have still (a), which is valid for *any* vector $\mathbf{r}$ as operand, *i.e.*

$$\Omega\epsilon\mathbf{r} = \epsilon_2\epsilon_1\mathbf{r} + \frac{\gamma_1\gamma_2}{c^2}\mathbf{v}_2(\mathbf{v}_1\mathbf{r}). \qquad (a)$$

As to ϵ, the reader may verify that none of the above four equations is contradicted by assuming it to be a longitudinal stretcher corresponding to $\mathbf{v}$, *i.e.* by writing, for any $\mathbf{r}$,

$$\epsilon\mathbf{r} = \mathbf{r} + \frac{\gamma - 1}{v^2}\mathbf{v}(\mathbf{vr}).$$

Then Ω will be determined by (a). In fact, take for $\mathbf{r}$ a vector $\mathbf{n}$, normal to the plane $\mathbf{v}_1$, $\mathbf{v}_2$, and consequently normal also to $\mathbf{v}$ (which is always coplanar with $\mathbf{v}_1$, $\mathbf{v}_2$). Then $(\mathbf{v}_1\mathbf{n})$ and $(\mathbf{vn})$ will vanish, and $\epsilon\mathbf{n} = \mathbf{n}$, so that ($a$) will become

$$\Omega\mathbf{n} = \epsilon_2\epsilon_1\mathbf{n},$$

and since ϵ_1, ϵ_2 are longitudinal stretchers and $\mathbf{n}$ is normal to the axes of both,

$$\Omega\mathbf{n} = \mathbf{n}. \qquad (8)$$

Thus, the axis of rotation, or simply the axis of Ω, is normal to the plane $\mathbf{v}_1$, $\mathbf{v}_2$, while the angle of rotation is given by (7). The outstanding determination of the sense of rotation is left to the reader. To resume:

> The general or six-parametric Lorentz transformations $L(\mathbf{v}, \Omega)$ constitute a group, but the special or three-parametric transformations $L(\mathbf{v}, 1)$ or $L(\mathbf{v})$ *do not constitute a group*, though they contain the subgroups for parallel velocities. The successive application of two special Lorentz transformations with *non-parallel velocities* $\mathbf{v}_1$, $\mathbf{v}_2$ *gives always an* $L(\mathbf{v}, \Omega)$, that is to say, it is equivalent to a special Lorentz transformation *followed by a pure space-rotation* round an axis normal to $\mathbf{v}_1$ and $\mathbf{v}_2$, which turns $\mathbf{v} = \mathbf{v}_1 \mathbin{\#} \mathbf{v}_2$ into $\mathbf{v}_2 \mathbin{\#} \mathbf{v}_1$,—the former of these vectors being given by (1), and the latter by (2).

The above properties might be elegantly expressed in quaternionic language, by taking instead of our $Q[\quad]Q$ the more general operator $a[\quad]b$, consisting of a pair of unit quaternions a, b, whose

axes are not parallel. But this subject need not further detain us here.

We have touched the six-parametric Lorentz group only to elucidate the question of successive transformations, as intimately connected with the composition of velocities. But henceforth we shall hardly need it any more. In fact, our previous transformation $L(\mathbf{v})$, without any rotation of the space-framework, will be found sufficient for all physical purposes.

Having got through this, let us return to the 'Addition Theorem' of velocities, (1), with the purpose of illustrating its meaning by a few remarks and some simple examples.

In the first place, if both $\mathbf{v}_1$ and $\mathbf{v}_2$ are *small* as compared with the velocity of light, then, if magnitudes of second order are neglected, (1) reduces at once to

$$\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2=\mathbf{v}_2+\mathbf{v}_1,$$

which is the Newtonian or classical formula for the composition of velocities.

Next, consider the simplest case of *parallel* velocities. Then $\epsilon_1\mathbf{v}_2=\gamma_1\mathbf{v}_2$, and, as in Chap. V.,

$$\mathbf{v}=\frac{\mathbf{v}_1+\mathbf{v}_2}{1+(\mathbf{v}_1\mathbf{v}_2)/c^2},$$

or, counting the resultant velocity positively along $\mathbf{v}_1$,

$$v=\frac{v_1\pm v_2}{1\pm v_1v_2/c^2},$$

according as $\mathbf{v}_2$ is concurrent with or against $\mathbf{v}_1$. It will be enough to consider the former case, for which

$$v=\frac{v_1+v_2}{1+v_1v_2/c^2}. \tag{9}$$

Let both v_1 and v_2 be smaller than c, say, $v_1=c-m$, $v_2=c-n$, where m, n are positive and smaller than c. Then

$$v=\frac{(2c-m-n)c}{2c-m-n+mn/c}<c,$$

i.e. the resultant of any velocities smaller than the velocity of light in vacuo is again smaller than the velocity of light. In other words, c plays the part of an infinite velocity, inasmuch as it cannot be

obtained by the accumulation of any number of velocities smaller than c. This property, proved here for concurrent velocities, will be expected to hold *a fortiori* for velocities of any direction. The rigorous proof, to be based upon the general formula (4), is left to the reader as a useful and interesting exercise.

Again, if one of the compounded velocities, say v_1, is equal c, then, by (9),

$$v = \frac{c + v_2}{1 + v_2/c} = c,$$

i.e. the resultant of c and of any other parallel velocity (no matter whether it is smaller or equal to or even greater than c) *is again the velocity of light c.** This result becomes obvious, when it is remembered that in the present case the system S' becomes a *flatland*, perpendicular to the direction of motion, and that v_2 or the former p' is the velocity of our particle relative to S'. The whole path of the particle appears to the S-observers as a single point of that flatland, so that, for these observers, the particle might as well be fixed in S'.

The following is one of the most beautiful applications of Relativity that were made in the early times of the doctrine.

To emphasize better the meaning of the various velocities, write again, for the moment, p, v, p' instead of v, v_1, v_2, so that

$$p = \frac{v + p'}{1 + vp'/c^2}. \tag{9a}$$

Now, this can be put into the form

$$p = p' + \kappa v,$$

where κ, expressing the fraction of v, which is added to p', is given rigorously by

$$\kappa = \frac{1 - p'^2/c^2}{1 + vp'/c^2}, \tag{10}$$

and approximately, for moderate values of p'/c and *small values of v/c*, by

$$\kappa = 1 - \left(\frac{p'}{c}\right)^2. \tag{11}$$

* The discussion of cases of non-parallel velocities, to be based upon (4), is recommended to the reader.

Here p' is the velocity, as observed in S', of what we have hitherto called a 'material particle.' But in doing so, we have assumed only that it is something that can be recognized and watched in its changing position. Its being 'material' or not, mattered, in fact, but little. We might as well have spoken from the beginning of any comparatively permanent complex of sense-data, distinctly localizable in the S- and S'-spaces. Thus, if p' be the velocity of propagation or transfer of anything that can be watched,* from the S'-standpoint, and if v be the velocity of S' relative to S, then p, as given by (9*a*), will be the corresponding velocity of propagation or transfer, from the S-point of view, and the above κ will be the *dragging coefficient* of S' (if it be empty except for the framework), or, as the case may be, of the bodies or media carried along with S'. If, for example, S' is attached to a column of air blowing uniformly past an observer resting on earth (S), and if p' be the velocity of sound relative to S' (and consequently, by the principle of relativity, also the velocity of sound as would be obtained by our S-observer in quiet air), then (11) will be the dragging coefficient of air for sound. In this case p'/c is of the order $3\cdot 3 \,.\, 10^4/3 \,.\, 10^{10} \doteqdot 10^{-6}$, so that κ differs from unity by little more than one millionth, and we have a sensibly (though not rigorously) full drag of sound by air. Similarly, for light† propagated along a column of flowing water, as in Fizeau's experiment, if p' be its velocity relative to the water and taken from the S'-standpoint (and hence also the velocity of light in stationary water from the standpoint of an ordinary or S-observer), formula (11) will express the drag of light by water.

* 'Propagation,' as here defined, does not necessarily involve any material medium as the 'substratum' of the thing to be recognized and watched in its migrations, the only requirement being the possibility of its being watched so. Thus, we may have 'propagation' of a distortion along a rope, or of sound waves in air, or of electromagnetic 'disturbances' through empty space as well as through glass or water. The process of detecting and watching the waves or disturbances may be immediate in some, and very indirect in other cases, but this does not bring in any essential differences.

† In this case we can imagine an irregular train of light waves or a solitary wave or a sufficiently thin electromagnetic sheet which can be watched, at least theoretically. And if we wish we can reduce this case to that of the motion of a 'material particle,' by placing such a particle (in our imagination, of course) in that sheet and by requiring it to be permanently illuminated; for then it will have to move just as quickly as light in the medium in question. This is Laue's device, slightly modified. But I do not think that such a reduction to the motion of something tangible is seriously needed.

The only difference is that in this case the value of p'/c is no longer exceedingly small as for sound and air, and this is why the case is of considerable physical importance. For water in ordinary conditions p'/c is as great as $3/4$, and it approaches unity even more nearly for optically 'rarer' media. Generally, if n be the corresponding index of refraction, we have $p'/c = 1/n$, so that (11) gives at once

$$\kappa = 1 - \frac{1}{n^2},$$

and this is the famous *dragging coefficient of Fresnel*, which occupied so much of our attention in the early part of this volume, and which was found to be in such good agreement with experiment.

Thus, Fresnel's formula, which on the ground of the electron theory appeared as the outcome of a rather complicated play of minute particles, follows here as a simple consequence of the fundamental theorem of relativistic kinematics, quite independently of any theory of the structure of matter.

Notice that the above is but an approximate value of the dragging coefficient, and that its rigorous value would be, by (10),

$$\kappa = \frac{1 - 1/n^2}{1 + \beta/n}, \tag{12}$$

where $\beta = v/c$. But for the present Fresnel's formula, considering the technical difficulties of the measurements, is more than sufficiently accurate. Remember that in Fizeau's experiment, as repeated in an improved form by Michelson and Morley (p. 41), the water was flowing with a velocity of 8 metres per second, so that β was of the order 10^{-8}, while the observed value of the drag could be trusted to hardly more than two decimal figures. I do not know what possibilities lie in canal rays. At any rate the experimental discrimination between (12) and the Fresnel formula is a problem reserved for the future.

As a further example of composition of velocities, let us consider the case of *perpendicular* components. Returning once more to the notation adopted in the general formula (1), we have in the present case $(\mathbf{v}_1\mathbf{v}_2) = 0$ and $\epsilon_1\mathbf{v}_2 = \mathbf{v}_2$, so that the resultant $\mathbf{v} = \mathbf{v}_{12}$ of $\mathbf{v}_1$ followed by $\mathbf{v}_2$ becomes

$$\mathbf{v}_{12} = \mathbf{v}_1 \mathbin{\#} \mathbf{v}_2 = \mathbf{v}_1 + \frac{\mathbf{v}_2}{\gamma_1} = \mathbf{v}_1 + \mathbf{v}_2\sqrt{1 - \beta_1^2}. \tag{13}$$

Similarly, the resultant of $\mathbf{v}_2$ followed by $\mathbf{v}_1$ will be

$$\mathbf{v}_{21}=\mathbf{v}_2 \mathbin{+\!\!+} \quad =\mathbf{v}_2+\frac{\mathbf{v}_1}{\gamma_2}=\mathbf{v}_2+\mathbf{v}_1\sqrt{1-\beta_2^{\,2}}. \tag{14}$$

In Fig. 14, in which *OANB* is a rectangle, the former of these vectors is given, in absolute value and direction, by *OC*, and the latter by *OD*, while the diagonal *ON* represents the Newtonian resultant. As was already remarked, the absolute values of the

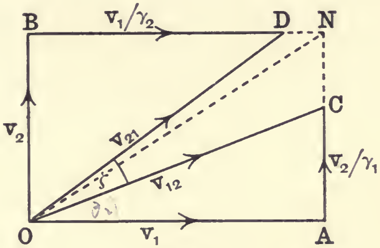

FIG 14.

relativistic resultants $\mathbf{v}_{12}$, $\mathbf{v}_{21}$ are equal to one another, the square of each being in the present case given by

$$v^2=v_1^{\,2}+v_2^{\,2}-\frac{1}{c^2}v_1^{\,2}v_2^{\,2}, \tag{15}$$

instead of which we may conveniently write

$$\beta^2=\beta_1^{\,2}+\beta_2^{\,2}-\beta_1^{\,2}\beta_2^{\,2},$$

or also, as a particular case of (*b*), p. 169,

$$\gamma=\gamma_1\gamma_2. \tag{16}$$

To obtain the angle $\zeta = COD$ enclosed by the two resultants, take their scalar product and divide it by v^2. The result will be

$$\cos\zeta=\frac{\gamma_1\beta_1^{\,2}+\gamma_2\beta_2^{\,2}}{\gamma\beta^2}. \tag{17}$$

Thus, for $v_1=v_2$ equal $\frac{1}{5}$, $\frac{1}{2}$, $\frac{9}{10}$ of the velocity of light, the angle ζ would be, in round figures, 1°, 8°, 43° respectively, or more accurately 1° 10′, 8° 13′, 42° 54′.

To use Sommerfeld's illustration,* if we have a rectangular ruler, whose edges coincide initially with *OA* and *OB* (Fig. 14), and,

* A. Sommerfeld, *Verhandlungen der Deutschen Phys. Ges.*, XI. 1909, p. 577.

while it is moved relatively to the paper (S) horizontally with the velocity v_1, the point of a pencil is led along the vertical edge with the velocity v_2 relative to the ruler (S'), then the pencil will draw the line OC, *e.g.* the segment $\overline{OC}$ in unit time (S-time). On the other hand, if the ruler is moved vertically with velocity v_2 and the pencil is led along its horizontal edge with velocity v_1, the point of the pencil will draw the line OD. According to classical kinematics, the line drawn would be in both cases the diagonal of the rectangle. Notice that from the paper-standpoint the velocities to be compounded are: in the first case OA and AC (not AN), and in the second case OB and BD (not BN). In the old kinematics there was no question of discriminating between the paper- and the ruler-standpoints.

So much to explain the true meaning of $\mathbf{v}_1 \# \mathbf{v}_2$, as distinguished from $\mathbf{v}_2 \# \mathbf{v}_1$.

The space in the ordinary sense of the word, or the space of positions being assumed Euclidean in both the old and the new theory, the space representative of velocities, or what is shortly called the *kinematic space*, is again the Euclidean space in classical kinematics, but non-Euclidean in relativistic kinematics. In order to represent the resultant $\mathbf{v}_{12}$ on the same Euclidean plane drawing with the component velocities, we had to cut off from $\mathbf{v}_2$ the piece CN, and similarly, in constructing $\mathbf{v}_{21}$ we had to cut off from $\mathbf{v}_1$ the piece DN. If we want to obtain the resultant by a triangle construction without cutting off anything from the segments representing the component velocities or any functions of each of these velocities alone, then we have to use a non-Euclidean space, *e.g.* Lobatchewsky's and Bolyai's space of constant negative curvature, or, as it is appropriately called, a hyperbolic space.*

In short, the relativistic kinematic space is a **hyperbolic** space.

* This was first pointed out explicitly by V. Varićak, *Phys. Zeitschrift*, Vol. XI. 1910, pp. 93, 287, 586; cf. also *Jahresbericht der deutschen Math. Vereinigung*, Vol. XXI. 1912, p. 103, where all his contributions to the subject are collected. But it must be noticed that materially the discovery was made previously, in 1909, by Sommerfeld (*Verh. deutsch. Phys. Ges.*, XI. p. 577), when he proved that the relativistic formulae for the composition of velocities are '*no longer the formulae of plane but those of spherical trigonometry* (with imaginary sides),' *e.g.* those which are obtained from the usual ones by replacing the real radius R of the sphere by ιR, —and the identity of these formulae with those valid for triangles in Lobatchewskyan space has been well known for a long time. In fact, this identity was pointed out by Lobatchewsky himself.

To see this, take, for simplicity, the above case of $\mathbf{v}_1 \perp \mathbf{v}_2$. Denote the angle contained between $\mathbf{v}_1$ and the resultant $\mathbf{v}=\mathbf{v}_{12}$, *i.e.* the angle AOC of Fig. 14, by θ_2. Then, by (13),

$$\tan\theta_2 = \frac{v_2}{v_1\gamma_1} = \frac{\beta_2}{\beta_1\gamma_1}$$

and, by (16),
$$\gamma = \gamma_1\gamma_2.$$

Now, instead of the absolute value of each of the velocities, introduce the corresponding imaginary angle ω,

$$\omega = \arctan(\iota\beta),$$

as defined by (10), Chap. V. Then $\gamma=\cos\omega$, $\beta\gamma = -\iota\sin\omega$, and the above pair of formulae will become

$$\cos\omega = \cos\omega_1 \,.\, \cos\omega_2,$$

$$\tan\theta_2 = \frac{\tan\omega_2}{\sin\omega_1},$$

and these are the known formulae of spherical trigonometry for a right-angled triangle, whose sides and hypothenuse are ω_1, ω_2, ω and whose angle opposite to ω_2 is θ_2, the only difference being that here all the sides are imaginary. This is the property remarked by Sommerfeld (cf. last footnote).

Next, to get rid of the imaginary sides, introduce, for each velocity, instead of ω the *real angle* a, as defined by (20), Chap. V., such that

$$\tanh a = \beta = v/c. \tag{18}$$

Then, as was previously noticed, $\omega = \iota a$, and, since

$$\sin(\iota a) = \iota\sinh a, \quad \cos(\iota a) = \cosh a,$$

the above formulae become at once

$$\left.\begin{aligned} \cosh a &= \cosh a_1 \,.\, \cosh a_2 \\ \tan\theta_2 &= \frac{\tanh a_2}{\sinh a_1}. \end{aligned}\right\} \tag{19}$$

Now, these are exactly the formulae for a right-angled triangle in Lobatchewskyan or hyperbolic space.* Thus, if a_1 and a_2 (Fig. 15)

* Cf. N. I. Lobatchewsky's *Zwei geometrische Abhandlungen*, translated from Russian into German and edited by F. Engel, Leipzig, 1898. Also 'Non-Euclidean Geometry,' by Frederick S. Woods, in *Monographs on Topics of Modern Mathematics, etc.*, London, 1911, or R. Bonola's *Non-Euclidean Geometry*, translated by H. S. Carslaw, Chicago, 1912.

are segments of geodesics or shortest lines in hyperbolic space, representing the component velocities, the shortest line a, completing the triangle, will represent the resultant velocity, as regards both size and inclination, θ_2. The same property may be proved to hold in general, *i.e.* for component velocities including with one another any angle. Here it will be enough to give the length of a.

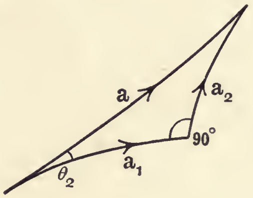

FIG. 15

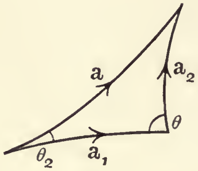

FIG. 16.

Denoting by $\pi - \theta$ the angle $\mathbf{v}_1$, $\mathbf{v}_2$, so that θ itself is the angle opposite to a (Fig. 16), we have

$$\frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2) = -\beta_1\beta_2 \cos\theta,$$

so that our previous formula (b),

$$\gamma = \gamma_1\gamma_2\left[1 + \frac{1}{c^2}(\mathbf{v}_1\mathbf{v}_2)\right],$$

becomes at once

$$\cosh a = \cosh a_1 \,.\, \cosh a_2 - \sinh a_1 \,.\, \sinh a_2 \,.\, \cos\theta. \qquad (20)$$

The determination of the angle θ_2, by means of the general formula (1), is left to the reader.

Notice that, as long as we are concerned only with two velocities and their resultant, we have no need of three-dimensional hyperbolic space. What we want then is a Lobatchewskyan plane or a surface of constant negative curvature. Now this may be easily procured of any size in Euclidean space. Models of such a surface, known as a *pseudosphere*, which is a surface of revolution,* belong now to the outfit of many mathematical class-rooms. Our last two figures must be imagined to be drawn on a pseudosphere (which certainly has nothing more imaginary about it than the page on which Figs. 15 and 16 are drawn), the curved sides of our drawings being as straight as possible on such a surface. Thus, having at our disposition a pseudosphere, we could study at our leisure the non-

* See, for instance, Bonola's book, just quoted, p. 132.

commutativity and all the remaining properties of the addition of velocities. In this way the relativistic rules of the composition of velocities could be made accessible even to all those who do not like to think of hyperbolic, and other non-Euclidean, spaces.

It has been proposed by Dr. Robb* to call our above a, as defined by (18), that is

$$a = \text{arc tanh}\,\frac{v}{c}\,, \tag{21}$$

the **rapidity**, corresponding to the velocity v. It seems a very convenient name for the purpose. Using it, we may briefly restate the above result as follows:

Any two *rapidities* are compounded by the triangle-rule in *hyperbolic* space.

Whence also: the resultant of any number of rapidities arranged in *a chain* in hyperbolic space, is the geodesic or the straight line of that space, drawn from the beginning to the end of the chain.

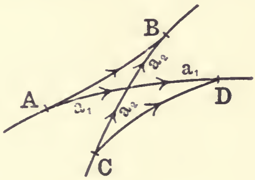

FIG. 17.

Notice that if rapidity is to involve 'direction' as well as size or absolute value, it has to be considered as a vector *localized in its own line*, *i.e.* in a Lobatchewskyan straight line or shortest line upon our pseudosphere. In connexion with this we have only the *triangle*-rule, and not the parallelogram-rule, as in Newtonian kinematics. There are no parallelograms in hyperbolic space or upon a pseudosphere, any more than upon a sphere. To express that direction is involved, we may write for the rapidities $\mathbf{a}_1$, $\mathbf{a}_2$, etc., and use the ordinary sign + for their addition, keeping in mind that each of these rapidity-vectors can be shifted only along its own line, and, consequently, that their addition is non-commutative, unless $\mathbf{a}_1$, $\mathbf{a}_2$ are on the same line. Thus, the rapidity $\mathbf{a}_1 + \mathbf{a}_2$ (Fig. 17) is AB, while $\mathbf{a}_2 + \mathbf{a}_1$ is CD, which, though of the same length, is on a different line.

*Alfred A. Robb, *Optical Geometry of Motion*, Cambridge, W. Heffer & Sons, 1911.

Remembering that $\tanh a = (e^a - e^{-a})/(e^a + e^{-a})$, we can write, instead of (21),

$$a = \tfrac{1}{2} \log \frac{1+\beta}{1-\beta} = \beta + \tfrac{1}{3}\beta^3 + \tfrac{1}{5}\beta^5 + \ldots. \qquad (21a)$$

For small values of β we have, up to quantities of the second order, $a \doteq \beta = v/c$, so that for small velocities the corresponding rapidities are small fractions, of the order of β, and the Lobatchewskyan triangle becomes a Euclidean triangle, as in classical kinematics. It seems worth mentioning that to *unit* rapidity corresponds a huge velocity, amounting to $\frac{3}{4}$ of the velocity of light; more accurately, we have

$$\beta = \cdot 7616 \text{ for } a = 1.$$

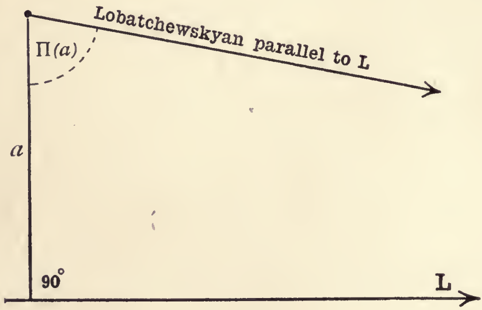

Fig. 18.

From (21a) we see most immediately that to the velocity of light itself corresponds *an infinite rapidity*,

$$a = \infty \quad \text{for} \quad \beta = 1.$$

Now, if two sides of a pseudospherical triangle are finite, its third side is also finite. Thus, our previous statement, that the resultant of any velocities smaller than that of light is again smaller than the velocity of light, is reduced to an obvious property of hyperbolic triangles.

To close the discussion of this beautiful subject, but one remark more. Lobatchewsky's $\Pi(a)$, the **angle of parallelism** for the length a, as explained by Fig. 18, is related to the above hyperbolic functions, for any a, as follows:

$$\sin \Pi(a) = \frac{1}{\cosh a}; \quad \cos \Pi(a) = \tanh a, \quad \tan \Pi(a) = \frac{1}{\sinh a}. \qquad (22)$$

Thus, equations (19) can be written, in terms of ordinary trigonometric functions of the respective angles of parallelism,

$$\left.\begin{aligned} \sin \Pi(a) &= \sin \Pi(a_1) \,.\, \sin \Pi(a_2) \\ \tan \theta_2 &= \tan \Pi(a_1) \,.\, \cos \Pi(a_2), \end{aligned}\right\} \tag{23}$$

which is the original form of Lobatchewsky's own formulae, for a right-angled triangle. Similarly, the general formula (20), will become

$$\sin \Pi(a) = \frac{\sin \Pi(a_1) \,.\, \sin \Pi(a_2)}{1 - \cos \Pi(a_1) \,.\, \cos \Pi(a_2) \,.\, \cos \theta}, \tag{24}$$

which is Lobatchewsky's fundamental formula. The unit of length here adopted is that employed by Lobatchewsky, *i.e.* that length whose negative reciprocal square is the curvature of the representative hyperbolic space, or the curvature of the pseudosphere upon which the triangles are to be drawn. Thus, if we take for that purpose a pseudosphere of curvature $-1/100$ cm^2., a segment of its geodesic 10 centimetres long will correspond to the rapidity $a = 1$, and consequently will represent the velocity $\cdot 76\, c$ which is a little above the velocity of light in water.

Instead of (18), we shall now have, by the second of (22), and omitting the unnecessary argument,

$$\cos \Pi = \frac{v}{c} = \beta. \tag{18a}$$

For very small values of β the angle of parallelism Π is nearly a right angle, as in a Euclidean plane. Thus, for the earth's orbital motion $\beta = 10^{-4}$ and $\Pi = 89^\circ\ 59'\ 39''\cdot 4$, so that the departure from Euclid amounts only to $20''\cdot 6$. But if we turn to swift electrons, as observed in kathode rays and β-rays of radioactive substances, the angle of parallelism is very considerably reduced. For $\beta = \cdot 90$ and $\cdot 95$ (Kaufmann observed even $\cdot 99$ and more) I find $\Pi = 25^\circ\ 50'$ and $18^\circ\ 12'$ respectively. At the limit, for light-velocity, the angle of parallelism would vanish altogether.

CHAPTER VII.

PHYSICAL QUATERNIONS. DYNAMICS OF A PARTICLE.

THE importance of the study of world-vectors or of physical quaternions for relativistic investigations is obvious. For, if the form of the laws of physical phenomena is to be preserved by the Lorentz transformation, they can involve besides the time and the coordinates, and, of course, besides any invariants, only such sets of magnitudes which, *caeteris paribus*, bear in any of the legitimate systems the same relations to its time and coordinates as in any other of such systems. Therefore, physical quaternions (or whatever mathematical form we may choose for tetrads of magnitudes transformed like l, x, y, z and of sets of magnitudes derived from them) constitute, as it were, the building material of the modern relativist. And what is most important to keep in mind, is that he cannot use any other material. For if he did, he would be sure to infringe against the fundamental principle of the whole theory.

To try to describe in a few abstract sentences the way how this material is procured and how it is used, would be a vain attempt. The reader will see it best from particular cases.

As yet we had, properly speaking, only one physical quaternion, which we made the standard of such quaternions, *e.g.* the position-quaternion

$$q = l + \mathbf{r} = \iota ct + \mathbf{r}. \tag{1}$$

This was transformed into q' by the operator $Q[\quad]Q$. If any quaternion X was transformed into X' by the same operator, we wrote $X \sim q$, and if it had also, like q, an imaginary scalar and a real vector, we wrote $X \simeq q$, and called X a physical quaternion. Such was our definition given in Chap. V., entirely equivalent to that of a four-vector.

Now let us look for other physical quaternions. An indefinite number of such can be obtained at once from q itself.

In fact, let q belong, say, to a material particle at a given instant t of its history. Let the particle move about in an arbitrary manner, and let $\mathbf{p}$ be its instantaneous velocity in S. Then its position-quaternion at the instant $t+dt$ will be $q+dq$, and this as well as q will certainly be a physical quaternion. And since $Q[\quad]Q$ is distributive (or since the Lorentz transformation is linear and homogeneous), the difference of these two quaternions, *i.e.*

$$dq = dl + d\mathbf{r} = [\iota c + \mathbf{p}]\,dt,$$

will again be a physical quaternion, $\simeq q$. Therefore, as we know from Chap. V., its tensor

$$\mathrm{T}dq = \iota\, dt\sqrt{c^2 - p^2}$$

will be an invariant. Divide it by ιc; then

$$d\tau = dt\sqrt{1 - \frac{p^2}{c^2}} = \frac{dt}{\gamma_p} \tag{2}$$

will again be an *invariant*. Its value will be real, provided that p is not greater than c. And since dq is a physical quaternion, we shall have also

$$Y = \frac{dq}{d\tau} \simeq q, \tag{3}$$

that is, Y will again be a physical quaternion. Let us call it the **velocity-quaternion** of the particle in question. Its developed form is

$$Y = \gamma_p[\iota c + \mathbf{p}], \tag{3a}$$

where $\mathbf{p}$ is the ordinary vector-velocity of the particle, justifying the above name.

The plain meaning of our result is that $Y' = QYQ$, *i.e.* that

$$\iota c\gamma_p \quad \text{and} \quad \mathbf{p}\gamma_p$$

are transformed as l and $\mathbf{r}$, or, what is the same thing, that

are transformed like
$$\begin{matrix} \gamma_p & \text{and} & \mathbf{p}\gamma_p \\ t & \text{and} & \mathbf{r}. \end{matrix}$$

Using this, the reader will obtain at once the addition theorem of velocities, identical with (1*a*), Chap. VI., along with the formula

$$\gamma_p = \gamma_v \gamma_{p'} \left[1 + \frac{1}{c^2} (\mathbf{v}\mathbf{p}') \right]$$

(identical with (*b*), p. 169), which is a consequence of that theorem. Thus, the relativistic rule for the composition of velocities is implied in the statement that Y is a physical quaternion.

The infinitesimal scalar $d\tau$, as defined by (2), deserves special attention. For $p=0$ it reduces to dt, the element of ordinary S-time, but is, in general, smaller than dt. It has the advantage of being an invariant, which dt is not. In other words, the value of $d\tau$ is independent of the choice of our standpoint, being equal for all legitimate systems. It belongs to the particle. The same property will obviously hold for

$$\tau = \int d\tau = \int \frac{dt}{\gamma_p},$$

where the integral is taken along any portion of the particle's history, or along any segment of its world-line, from an arbitrarily fixed initial point to the variable end-point. The parameter τ, thus defined, may be called, after Minkowski, **the proper time*** of the particle. The velocity $\mathbf{p}$ of the particle, entering into each element of τ by its square, may, in general, vary from instant to instant, as regards both absolute value and direction. If the particle is fixed in S, its proper time is the ordinary time t of the system S. And if the particle moves uniformly in S, we can imagine a system S' in which it will be at rest. And then the proper time of the particle will become the ordinary time of that system.

The velocity-quaternion may now be described as the derivative of the position-quaternion with respect to the proper time of the particle. It will often be convenient to use the dot for this differentiation. Thus, $Y = \dot{q}$.

The name corresponding to Y in the language of four-dimensional algebra would be *four-velocity*,† and its matrix-form would be simply, by (3*a*),

$$\gamma_p | p_x, p_y, p_z, \iota c |.$$

* *Eigenzeit.*

† Minkowski's *Bewegungsvektor*, Laue's *Vierergeschwindigkeit.*

Remember that $d\tau$, as originally defined, was simply the tensor of dq divided by ιc. The tensor of the velocity-quaternion is, therefore,

$$\mathrm{T}\,Y = \iota c. \tag{4}$$

We know, from Chap. V., that the tensor of every physical quaternion is an invariant. In the present case this knowledge does not furnish us anything new. For c is, by the fundamental assumptions of the theory, a universal constant. The norm of Y being negative, namely equal to $-c^2$, the velocity-quaternion is always *time-like*. In Minkowski's language we should say that the four-velocity is along the world-line of the particle in question.

Since Y is a physical quaternion and τ is an invariant,

$$Z = \frac{dY}{d\tau} = \ddot{q} \tag{5}$$

will again be a physical quaternion which, for obvious reasons, may be called the **acceleration-quaternion.** So also will $d^3q/d\tau^3$, etc. be physical quaternions, each $\simeq q$, and obviously also $d^3q_c/d\tau^3$, etc., each $\simeq q_c$. But of all these derivatives of q we shall hardly need more than the first two, containing the velocity and the acceleration.

Let Y_c be the conjugate of Y. Then, by Quat. 7, we can write for its norm the product YY_c or also SYY_c, and consequently, instead of (4),

$$YY_c = -c^2. \tag{4a}$$

Differentiating this with respect to τ, we have

$$ZY_c + YZ_c = 0, \tag{6a}$$

or also

$$SZY_c = 0, \tag{6}$$

which says precisely the same thing as (6*a*).* Such then is the relation which holds always between the acceleration- and the velocity-quaternion of a particle. Using the developed form $q = l + \mathbf{r}$, we should have, correspondingly,

$$(\dot{\mathbf{r}}\dot{\mathbf{r}}) + \dot{l}^2 = -c^2 \tag{4b}$$

and

$$(\dot{\mathbf{r}}\ddot{\mathbf{r}}) + \dot{l}\,\ddot{l} = 0, \tag{6b}$$

* In fact, the reader will find at once that, for any pair of quaternions a, b,

$$ab_c + ba_c = 2Sab_c = 2Sa_cb.$$

or, in a still more developed form,

$$\dot{x}^2+\dot{y}^2+\dot{z}^2+\dot{l}^2=-c^2 \tag{4c}$$

and

$$\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}+\dot{l}\ddot{l}=0. \tag{6c}$$

In four-dimensional language, as explained in Chap. V., the last formula would read: *The four-acceleration is always normal to the four-velocity* and, consequently, to the world-line of the particle,—a famous statement of Minkowski. This cardinal property finds then its short quaternionic expression in (6). Observe that the left side of that equation is the same thing as Sommerfeld's scalar product of the corresponding four-vectors. But the invariance of such expressions is seen more immediately on the quaternionic scheme. In fact, remembering that $QQ_c=Q_cQ=1$, we have, by Quat. 6,

$$\mathrm{S}Z'Y'_c=\mathrm{S}QZQQ_cY_cQ_c=\mathrm{S}QZY_cQ_c=\mathrm{S}Q_cQZY_c=\mathrm{S}ZY_c.$$

Next, as regards the transformational properties of the acceleration. These are entirely determined by saying that $Z=\iota c\ddot{t}+\ddot{\mathbf{r}}$ is a physical quaternion. For this means simply that $\ddot{t}$, $\ddot{\mathbf{r}}$ are transformed like t, $\mathbf{r}$. If, therefore, S' be a system moving relatively to S with the uniform velocity $\mathbf{v}$, we have, according to $(1'b)$, Chap. V.,

$$\left.\begin{aligned}\ddot{\mathbf{r}}&=\epsilon_{\mathbf{v}}\ddot{\mathbf{r}}'+\mathbf{v}\gamma_v\ddot{t}'\\ \ddot{t}&=\gamma_v[\ddot{t}'+\frac{1}{c^2}(\mathbf{v}\ddot{\mathbf{r}}')],\end{aligned}\right\} \tag{7}$$

where the subscripts are to remind us that γ, ϵ are to be taken for the velocity $\mathbf{v}$. The dots denote, on both sides, differentiation with respect to the same variable τ. For, as the reader already knows, $d\tau'=d\tau$. There is no difficulty in developing these formulae and thus finding the ordinary acceleration

$$\mathbf{a}=\frac{d\mathbf{p}}{dt}$$

in terms of $\mathbf{a}'=d\mathbf{p}'/dt'$ and $\mathbf{p}'$ (or *vice versa*), for any pair of legitimate systems S, S' picked out at random. But this would hardly be worth the trouble.

To see the plain kinematical meaning of the second derivatives with respect to τ and hence of the whole acceleration-quaternion, we have to place ourselves at a standpoint which bears the simplest

possible relation to the moving particle itself. Let us then take for S' that particular system of reference with respect to which the particle is instantaneously at rest. In other words, let S' be a system whose uniform velocity $\mathbf{v}$, relative to S, is equal in size and direction to the instantaneous velocity of the particle, *i.e.* to the value of $\mathbf{p}$ at a given instant of its history. Then, at that instant, $p'=0$ and $\gamma'=\gamma(p')=1$. Now, we had, generally, $dt/d\tau=\gamma(p)$. Therefore,

$$\ddot{t}'=\gamma'\frac{d\gamma'}{dt'}=\frac{d\gamma'}{dt'}=\frac{1}{c^2}\gamma'^3p'\frac{dp'}{dt'}=0,$$

or $\ddot{l}'=0$, as might have been expected, and in a similar way,

$$\ddot{\mathbf{r}}'=\frac{d\gamma'\mathbf{p}'}{dt'}=\frac{d\mathbf{p}'}{dt'}=\mathbf{a}',$$

so that $Z'=\ddot{l}'+\ddot{\mathbf{r}}'$, the acceleration-quaternion relative to S', for the instant in question, is simply

$$Z'=\mathbf{a}', \tag{8}$$

i.e. equal to the ordinary acceleration of the particle with respect to S'. Since S' is that particular system of reference in which the particle is instantaneously at rest, it may be called the *rest*-system and the corresponding $\mathbf{a}'$ **the rest-acceleration** of the particle.*

Thus, the scalar part of the acceleration-quaternion Z' vanishes identically, and its vector part is equal to the rest-acceleration.† Consequently, $\mathrm{T}Z'=a'$. And since the tensor of every physical quaternion is an invariant, we have also, for any legitimate system S,

$$\mathrm{T}Z=a'. \tag{9}$$

In words, *the tensor of the acceleration-quaternion is equal to the absolute value of the rest-acceleration of the particle.* It acquires thus an immediate kinematical meaning. At the same time formulae (7), in which we have now to write $\mathbf{v}=\mathbf{p}$, give us, for the system S which in a certain sense is an unnatural system of reference,

$$\ddot{\mathbf{r}}=\epsilon\mathbf{a}' \tag{10}$$

* In German, *Ruh*beschleunigung.

† This result could be foreseen. In fact, the 'ordinary' time of our particular S' coincides, in its element in question, with the proper time of the particle.

and $\ddot{t}=c^{-2}\gamma(\mathbf{pa}')$, so that the whole acceleration-quaternion may be written:

$$Z=\frac{dY}{d\tau}=\frac{\iota}{c}\gamma(\mathbf{pa}')+\epsilon\mathbf{a}'. \tag{11}$$

Here, $\mathbf{p}$ is the velocity of the particle relative to S; $\gamma=\gamma_p$, and the stretcher $\epsilon=\epsilon_{\mathbf{p}}$, of ratio γ_p, acts along the instantaneous direction of $\mathbf{p}$ or tangentially to the path of the particle. Thus, in Cartesians, if the tangent to the path of the particle, drawn in the direction of its motion, be our instantaneous x-axis,

$$\ddot{x}=\gamma a_x', \quad \ddot{y}=a_y', \quad \ddot{z}=a_z' \tag{10a}$$

and $c\ddot{t}=\beta\gamma a_x'$. If the y-axis be taken in the osculating plane of the path, then $\ddot{z}=0$. Since we already know, by (9), that

$$\ddot{x}^2+\ddot{y}^2+\ddot{z}^2-c^2\ddot{t}^2=a'^2,$$

the formula for $\ddot{t}$ becomes superfluous.

Finally, to express $\ddot{\mathbf{r}}=d^2\mathbf{r}/d\tau^2$ in terms of the ordinary S-acceleration $\mathbf{a}=d\mathbf{p}/dt$, remember once more that $dt/d\tau=\gamma$. Since, by the definition of γ,

$$\frac{d\gamma}{dt}=\frac{1}{c^2}\gamma^3 p\frac{dp}{dt}=\frac{1}{c^2}\gamma^3(\mathbf{pa}),$$

the result will be

$$\ddot{\mathbf{r}}=\gamma\frac{d\gamma\mathbf{p}}{dt}=\gamma^2\left[\mathbf{a}+\frac{1}{c^2}\gamma^2\mathbf{p}(\mathbf{pa})\right]=\gamma^2\left[\mathbf{a}+\beta^2\gamma^2\mathbf{u}(\mathbf{ua})\right],$$

where $\mathbf{u}$ is the unit of $\mathbf{p}$. Now, $1+\beta^2\gamma^2=\gamma^2$, identically. Therefore, the bracketed expression is the vector sum of the longitudinal part of $\mathbf{a}$ magnified γ^2 times and of its unaltered transversal part, or simply the result of a double application of the stretcher ϵ. Thus, ultimately,

$$\ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{d\tau^2}=\gamma^2\epsilon^2\mathbf{a}, \tag{12}$$

whence also, by (10),

$$\gamma^2\epsilon\mathbf{a}=\mathbf{a}', \tag{13}$$

giving the connexion between $\mathbf{a}$ and the rest-acceleration. Or, in Cartesians, with the above choice of axes, for the longitudinal and the transversal components of $\ddot{\mathbf{r}}$,

$$\ddot{x}=\gamma^4 a_x, \quad \ddot{y}=\gamma^2 a_y, \quad \ddot{z}=\gamma^2 a_z, \tag{12a}$$

and

$$\gamma^3 a_x=a_x', \quad \gamma^2 a_y=a_y', \quad \gamma^2 a_z=a_z'. \tag{13a}$$

By (13) we have also, writing $p/c=\beta$,

$$a\,.\,\gamma^3\sqrt{1-\beta^2\sin^2(\mathbf{p},\mathbf{a})}=a', \tag{14}$$

which is merely a developed form of (9). In fact, the left ~~right~~-hand side of (14) is seen, by (11), to be identical with TZ.

The simplest case of motion of a particle occurs when a' is permanently nil, and consequently also $a=0$. This is, as in classical kinematics, the trivial case of uniform rectilinear motion. Such motion preserves its character in all legitimate systems. In fact, owing to the linearity of the Lorentz transformation, any motion which is uniform and rectilinear with respect to one of these systems will be so relatively to any other of them. A straight world-line will remain straight. The next simplest kind of motion, which also preserves its character in all such systems of reference, occurs when the non-vanishing *rest-acceleration is constant in size and direction*, *i.e.* when $d\mathbf{a}'/d\tau'=0$, and hence also $d\mathbf{a}'/dt=0$. Then, by (13), the vector $\gamma^2\epsilon\mathbf{a}$ is constant in S, that is to say, independent of t. But since the axis of the stretcher ϵ, or the x-axis in (13a), instead of being fixed, is at every instant tangential to the path of the particle, which may be curvilinear, it does not follow that even the direction of the acceleration $\mathbf{a}$ will be constant in S. Thus, the general case of such motion, which is the counterpart of the uniformly accelerated or parabolic motion of classical kinematics, would still be fairly complicated. The simplest sub-case, which also will show best the characteristic properties of this kind of motion, occurs, of course, when the particle moves on a *straight line*. Let this be our x-axis. Then, by (13),

$$\gamma^3 a=\gamma^3\frac{dp}{dt}=a',$$

or

$$\frac{a'}{c}\,dt=\gamma^3\,d\beta=\frac{d\gamma}{\beta}=\frac{1}{2}\,\frac{d(\gamma^2)}{(\gamma^2-1)^{\frac{1}{2}}},$$

whence, counting the time t from the instant at which $p=0$,

$$p=\frac{dx}{dt}=a't\,.\left[1+\left(\frac{a't}{c}\right)^2\right]^{-\frac{1}{2}}. \tag{15}$$

Or we may write, equivalently,

$$\beta\gamma=\frac{a'}{c}\,t. \tag{15a}$$

Thus, as long as $a't$ is small in comparison with the velocity of light (whether before or after the instant when the particle was at rest in S), we have, approximately, $p \doteq a't$, and $x = \frac{1}{2}a't^2 + const.$, as in the Galilean free fall. But after a sufficiently long time the neglected terms begin to make themselves sensible, and the velocity of the particle, instead of increasing beyond all limits, tends asymptotically to the velocity of light. In fact, we have from (15), for any given a', $p = \mp c$ for $t \doteq \mp \infty$.

Integrating once more, and choosing the origin of x so that, for $t = 0$, $x = x_0 = c^2/a'$, we obtain

$$x^2 - c^2t^2 = \left(\frac{c^2}{a'}\right)^2. \tag{16}$$

Thus, the world-line of our particle, in rectilinear motion with constant rest-acceleration, is an equilateral hyperbola (Fig. 19), the

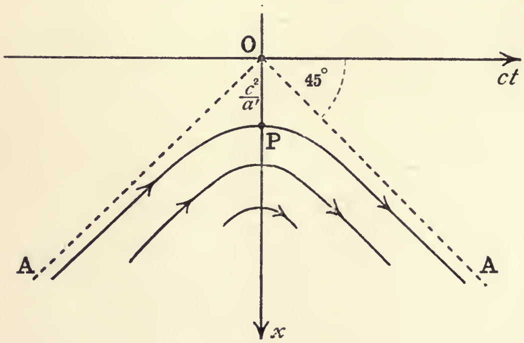

FIG. 19.

length of whose semiaxes is equal $\frac{c^2}{a'}$. This motion has therefore been called by Born, who was the first to study it, **hyperbolic motion.***
The asymptotes AO, OA correspond, as in a previous figure, to the velocity c, directed towards and away from the origin. The particle arrives from $x = \infty$ with light-velocity, moves up the x-axis with ever-diminishing velocity towards P, the vertex of the representative hyperbola, where its velocity is nil. Then it turns and moves down the x-axis with increasing velocity, which again tends asymptotically to the light-velocity. The larger the value of the rest-acceleration a', the more does the hyperbola penetrate into the angle AOA, and the

* In German, *Hyperbelbewegung*. M. Born, 'Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips,' *Ann. d. Physik*, Vol. XXX. 1909, p. 1.

more sudden is the passage of the particle's velocity from $-c$ through zero to $+c$. Taking, instead of S, another system of reference S'' which moves uniformly along the x-axis, and whose origin coincides with O at the instant $t''=t=0$, we shall have again equation (16) for the new variables. For, $x^2-c^2t^2$ is an invariant, and so is the acceleration a', by its very definition. It is this we meant by saying that the considered kind of motion preserves its character in different systems of reference,—a property which is not shared by the Galilean uniformly accelerated motion to which would correspond a parabola as world-line. Remember that in classical kinematics there was no question of discriminating between the ordinary S-acceleration a and the rest-acceleration a'.

We may mention here that the hyperbolic motion is particularly interesting in connexion with the theory of the relativistic 'rigid' body. But its chief importance lies herein that *any* variable motion can be closer approximated by it than by uniform motion. In other words, any curved world-line can be brought into closer contact with a hyperbola than with a straight line. There is for every point P of such a world-line a hyperbola of closest contact with the world-line, which plays the part of the familiar circle of curvature and which was called by Minkowski the *hyperbola of curvature*. If O be the centre of this hyperbola (whose vertex is at P), then the four-acceleration will be given by the world-vector drawn in the direction OP and having the absolute value $c^2/\overline{OP}$, or $\dfrac{c^2}{\text{semiaxis}}$. In fact, as we have just seen, the last expression is simply equal to a', and this again was seen to be identical with $\mathrm{T}Z$, or with the size of the four-acceleration which was always normal to the world-line. Remembering, on the other hand, that $\mathrm{T}Y=\iota c$, or that the square of the four-velocity is equal to $-c^2$, the reader will at once perceive the perfect analogy between the above property of $c^2/\overline{OP}$ and the familiar formula: normal acceleration = square of velocity divided by radius of curvature. It will also be noticed that, the square of the four-velocity being negative, the four-acceleration is directed away from the centre O of the osculating world-hyperbola, while in that more familiar case it is towards the centre of (the circle of) curvature of the path in three-dimensional space. But enough has now been said about the hyperbolic motion, in illustration of the use of the relativistic tetrads, Y and Z.

Such then are the properties of the velocity- and the acceleration-quaternion. These being simply the (first and second) derivatives of the position-quaternion q of a particle with respect to its proper time, our above considerations had a purely kinematical character. Although we have spoken of q as defining the position and the date of a 'particle,' the latter could mean anything which can be recognized at all and watched in its varying position. Of course, if this is to be possible, the 'particle' must have some or other characteristics of its own. But these must not necessarily be quantitatively measurable, to say nothing of their being constant in time or equal for different standpoints or systems of reference. The moving thing in question might have no such characteristic at all.

But let us suppose there is a certain magnitude of such a kind, that there is, more especially, a scalar coefficient belonging or attached to the particle and fulfilling the latter condition, *i.e.* *invariant* with respect to the Lorentz transformation. Denote it by m, without yet giving it any name. Then mq, $mY = m\dot{q}$, $mZ = m\dot{Y}$, and so on, will all be physical quaternions, and, consequently, each of them may be employed, along with other physical quaternions, for relativistic purposes, *i.e.* to write down laws of motion of the particle. Such laws would be admissible, in that sense of the word, that they would not infringe against the principle of relativity. But this does not imply, of course, that they will be obeyed by Nature. If such laws or equations are to be of any use for the physicist, and if they do not happen to cover an entirely unexplored ground, they have to coincide, roughly at least, and for ordinary circumstances, with what is otherwise known to hold in experience. In the present case we shall require that the relativistic equations of motion should coincide, approximately for small velocities, or rigorously, when referred to the rest-system, with Newton's second law of motion.

Keeping this in mind, let us see what are the consequences of assuming, as the equation of motion of our particle,

$$\frac{dmY}{d\tau} = X. \tag{17}$$

First of all, since the left-hand member is $\simeq q$, the right-hand member X, which is to be considered as a given function of position, and, in general, also of velocity and of time, must again be a physical quaternion. This fixes the transformational properties of

the quaternion X, and implies that it has an imaginary scalar and a real vector; the coefficient m being supposed real.

By its construction, equation (17) will preserve its form in all legitimate systems of reference.

Remembering that $dt/d\tau=\gamma$, write, instead of (17),

$$\frac{dmY}{dt}=\frac{1}{\gamma}X,$$

and denote the imaginary scalar of $\gamma^{-1}X$ by $\iota\nu$ and its real vector by $\mathbf{N}$, *i.e.* put

$$\frac{1}{\gamma}X=\iota\nu+\mathbf{N}. \tag{18}$$

Then (17) will split into the vector and the scalar equations

$$\left.\begin{aligned}\frac{d}{dt}m\gamma\mathbf{p}&=\mathbf{N}\\ \frac{d}{dt}mc\gamma&=\nu,\end{aligned}\right\} \tag{17a}$$

where $\mathbf{p}=d\mathbf{r}/dt$ is the ordinary velocity of the particle relative to S, and $\gamma=(1-p^2/c^2)^{-\frac{1}{2}}$.

Written for the *rest-system*, which we shall again denote by S', the first of these equations becomes at once

$$m\mathbf{a}'=m\frac{d\mathbf{p}'}{dt'}=\mathbf{N}',$$

i.e. identical with the classical equation of motion of a particle of mass m under the action of the impressed force $\mathbf{N}'$. Thus, the above requirement is fulfilled. In view of this property, the coefficient m is called the **rest-mass** of the particle.* The ordinary force, $\mathbf{N}'$ in the rest-system and, generally, $\mathbf{N}$ in any legitimate system S, is called the 'Newtonian force' in distinction from $\gamma\mathbf{N}$, the vector part of X, which is the 'Minkowskian force.' For reasons

* Lorentz, *Phys. Zeitschrift*, Vol. XI. 1910, calls m the 'Minkowskian mass,' and $\mathrm{V}X=\gamma\mathbf{N}$ the 'Minkowskian force,' since (17), with constant m, is the equivalent of the four equations of motion given by Minkowski; *Grundgleichungen*, Appendix, formulae (22). The four-vector corresponding to the whole quaternion X is Minkowski's '*moving force* (bewegende Kraft).' Einstein used the Newtonian force; his three equations of motion are identical with the first of (17*a*). Cf. Einstein's paper in *Jahrbuch der Radioaktivität und Elektronik*, Vol. IV. 1908, pp. 411-462, formulae (11).

which will appear when we come to consider the ponderomotive actions of the electromagnetic field, we shall have to consider the former and not the latter as *the* force acting upon the particle.

The second of (17a) becomes, for the rest-system,

$$\frac{dm}{dt'} = \frac{v'}{c}.$$

As will be seen in Chapter IX., there are reasons for admitting that even the rest-mass may vary with time. In fact, this will, in general, be the case when the internal state of the particle varies during its motion. But, to simplify matters, let us suppose that the particle's internal state is kept constant. Then its rest-mass m will be constant in time. This implies $v' = 0$, so that the whole quaternion X will be reduced, for the rest-system, to

$$X' = \mathbf{N}', \tag{18'}$$

and we shall have, for any legitimate system S,

$$\mathrm{T}X = \mathrm{T}X' = N', \tag{19}$$

where N' is the absolute value of the (Newtonian) force as estimated from the standpoint of the rest-system.

With this supposition of a *constant rest-mass*, equation (17) becomes

$$m\frac{dY}{d\tau} = mZ = X. \tag{20}$$

Now, by (6), $\mathrm{S}ZY_c = 0$, and consequently also

$$\mathrm{S}XY_c = 0, \tag{21}$$

or, in developed form, by (3a) and (18),

$$(\mathbf{Np}) = cv.$$

Hence, by the second of (17a), which is simply the scalar part of (20),

$$(\mathbf{Np}) = \frac{d}{dt}(mc^2\gamma). \tag{22}$$

Thus, $(\mathbf{Np})$ being the activity of the force $\mathbf{N}$, the scalar part of the quaternionic equation (20) expresses *the principle of energy*

(*principle of Vis-viva*), giving for the kinetic energy of the particle the value

$$T = mc^2(\gamma + \text{const.}).$$

If we require that, for $p = 0$ (*i.e.* for $\gamma = 1$), $T_0 = 0$, we have to put const. $= -1$. Ultimately, therefore, the **kinetic energy** of the particle, moving with the velocity $\mathbf{p}$ relative to S, becomes

$$T = mc^2(\gamma - 1) = mc^2\left[(1 - \beta^2)^{-\frac{1}{2}} - 1\right], \tag{23}$$

or, developed in a series,

$$T = \tfrac{1}{2}mp^2(1 + \tfrac{3}{4}\beta^2 + \tfrac{5}{8}\beta^4 + \ldots).$$

For small velocities, this reduces sensibly to the first term $\frac{1}{2}mp^2$, which is the classical value of the kinetic energy, since in this case the rest-mass becomes sensibly identical with its S-value.

The above expression of the kinetic energy was first given in Einstein's fundamental paper of 1905. An alternative, remarkable, form of (23), due to Minkowski, is

$$T = mc^2\frac{dt - d\tau}{d\tau}, \tag{23a}$$

and reads as follows: the kinetic energy of a particle, as estimated from the S-standpoint, is the product of its rest-mass by the square of the light-velocity and by the proportionate gain of the S-time with respect to the particle's proper time.

Let us now consider the vector part of the quaternionic equation of motion, or the first of (17*a*). This, which holds also for a variable rest-mass, may be read in the usual way: rate of change of momentum = force. Then the **momentum** of the particle, of rest-mass m, will be

$$\mathbf{G} = m\gamma\mathbf{p} = \frac{m}{\sqrt{1 - p^2/c^2}}\mathbf{p}. \tag{24}$$

Thus, to obtain the momentum we have to multiply the ordinary velocity $\mathbf{p}$ of the particle by $m\gamma$, and not by m. Some authors call, therefore, $m\gamma$ the 'ordinary mass' of the particle. But we have rather to avoid so many different names. It is quite sufficient to know that m, the rest-mass, enters in a certain way into the expression of momentum, and in a certain way into that of kinetic energy. The **momentum-quaternion**, which is always a physical quaternion, will simply be mY.

Next, to see the properties of m with respect to the ordinary acceleration $\mathbf{a}=d^2\mathbf{r}/dt^2$, return once more to the assumption of constant m, and write the first of (17a)

$$m\frac{d\dot{\mathbf{r}}}{dt}=\frac{m}{\gamma}\ddot{\mathbf{r}}=\mathbf{N}.$$

Then, by (12),

$$m\gamma\epsilon^2\mathbf{a}=\mathbf{N}, \tag{25}$$

where, it will be remembered, ϵ^2 is a stretcher of ratio γ^2 acting tangentially to the path. Thus, the force, though always contained in the osculating plane, will, in general, differ in direction from the acceleration. Instead of the old 'mass,' which was simply a scalar factor converting the acceleration $\mathbf{a}$ into the force $\mathbf{N}$, we have now the linear vector-operator

$$m\gamma\,.\,\epsilon^2.$$

Or, splitting the acceleration into its tangential and normal (or longitudinal and transversal) components, a_x, a_y,

$$m\gamma^3\,.\,a_x=N_x,\quad m\gamma\,.\,a_y=N_y. \tag{25a}$$

This result is expressed by saying that the particle has the **longitudinal mass**

$$m_l=m\gamma^3=\frac{m}{\sqrt{(1-\beta^2)^3}}, \tag{26a}$$

and the **transversal mass**

$$m_t=m\gamma=\frac{m}{\sqrt{1-\beta^2}}. \tag{26b}$$

For vanishing velocities both of these masses become identical with the rest-mass of the particle. With increasing velocity the longitudinal mass increases more rapidly than the transversal one. For $p=c$ both would become infinite. So also would the kinetic energy of the particle increase beyond all limits when the velocity of light is approached.

It is worth noticing here that the above m_l and m_t depend on the velocity of motion in exactly the same way as the longitudinal and the transversal electromagnetic masses of a Lorentz electron.* The

* In fact, the above formulae (26a), (26b) become identical with those of Lorentz when m is replaced by $\frac{e^2}{5\pi c^2 R}$ in the case of homogeneous volume-charge, and by $\frac{e^2}{6\pi c^2 R}$ in the case of homogeneous surface-charge, e being the charge and R the est-radius of the electron. Cf. Chap. VIII.

formula of Lorentz for the transversal electromagnetic mass is now fairly well verified by experiments on electrons constituting the β-rays. In the early stage of such experimental research other electronic formulae coincided equally well with the observed facts. It has been argued therefore that the *whole* mass of the electron is of purely electromagnetic origin. Now, the above relativistic formulae, giving the required dependence on velocity, have nothing electromagnetic about them. If, therefore, the doctrine of relativity is accepted, any part of the observed mass of the electron may be attributed to a non-electromagnetic origin. To obtain this we have only to give to the electron, instead of the usual 10^{-13} cm., a correspondingly greater radius, reducing thus its electromagnetic mass. Remember that what is given by observation is the total mass and the total charge of an electron, while its dimensions remain free, in very wide limits at least. But this subject cannot profitably be discussed here any further.

The longitudinal and the transversal masses of a particle, defined as the quotients of the corresponding components of force and acceleration, may also be written, by (24), in terms of the absolute value G of the momentum,

$$m_t = \frac{G}{p}, \quad m_l = \frac{dG}{dp}. \tag{27}$$

The first of these is simply (24) itself, and to see the truth of the second, we have only to remember that $d\gamma/dp = \gamma^3 p/c^2$.* The formulae (27) would even continue to be true if we had in the expression of the momentum, instead of the factor $m\gamma$, any other function of β alone, as the reader may easily prove for himself.

Let us once more return to the first of equations (17*a*), which may be written

$$\frac{d\mathbf{G}}{dt} = \mathbf{N}. \tag{28}$$

Multiply it on both sides vectorially by $\mathbf{r}$. Remember that the momentum coincides in direction with the velocity $\mathbf{p} = d\mathbf{r}/dt$, or that $\mathrm{V}\mathbf{pG} = 0$. Then the result will be

$$\frac{d}{dt}\mathrm{V}\mathbf{rG} = \mathrm{V}\mathbf{rN}. \tag{29}$$

* So that $dG/dp = m\gamma + mp\gamma^3 p/c^2 = m\gamma(1 + \beta^2\gamma^2) = m\gamma^3$.

In words: The rate of change of the **moment of momentum** is equal, in absolute value and direction, to the moment of the impressed force, both moments being taken about O, the origin of $\mathbf{r}$. This is the relativistic equivalent of what is known in classical dynamics as *the principle of areas.* The above moment of momentum is, in terms of the rest-mass m, and $\mathbf{r}$, $\dot{\mathbf{r}}$,

$$m\mathrm{V}\mathbf{r}\mathbf{G} = m\mathrm{V}\mathbf{r}\frac{d\mathbf{r}}{d\tau}.$$

In particular, if the moment $\mathrm{V}\mathbf{r}\mathbf{N}$ is permanently *nil*, *i.e.* if the impressed force is central, we have the equivalent of the principle of *conservation of areas*, that is, m being again supposed constant,

$$\mathrm{V}\mathbf{r}\frac{d\mathbf{r}}{d\tau} = \mathbf{A},$$

where the vector $\mathbf{A}$ is constant both in size and in direction, relative to the frame-work of reference S. In this case the particle moves in a plane, normal to $\mathbf{A}$, as it would also according to Newtonian mechanics. But there is the following difference. In terms of the usual polar coordinates r, θ, we have, by the last equation,

$$r^2\frac{d\theta}{d\tau} = A,$$

that is to say, equal areas swept out by the radius vector in equal intervals of the *proper time* of the particle, and not of the S-time. Using the time t of the observers fixed in S we should have

$$r^2\frac{d\theta}{dt} = A\sqrt{1 - p^2/c^2},$$

and this is variable, unless the particle happens to move uniformly along its orbit. Such then is the relativistic modification of Kepler's second law, valid for any central forces. For slow motion we fall back, of course, to the ordinary conservation of areas.

Leaving, for the present, any further dynamical questions, we shall close this chapter by developing some simple and general properties of certain combinations of physical quaternions, independent of their particular meaning. These will be found useful in connexion with the subject of electromagnetism to be treated in the next chapter. They may also have a certain interest of their own.

Let a, b, d, etc., be any physical quaternions, each $\simeq q$, and, consequently, a_c, b_c, d_c, etc., each $\simeq q_c$. What combinations of these

quaternions, obtained by their addition and multiplication, can be used for relativistic purposes, that is to say, for writing down equations which will satisfy the principle of relativity?

We need not dwell upon the sum $a+b+d+\ldots$ (or $a_c+b_c+d_c+\ldots$), which is again a physical quaternion, in the original sense of the word, and as such, is relativistically available. But having mentioned the sum at all, it may be good to observe that a sum of **antivariant** quaternions,* as, for example,

$$a+b_c,$$

cannot be used. For not only is this sum not covariant with q, nor with q_c, but, when subjected to the Lorentz transformation, it is split, the two addends being torn asunder, thus

$$a'+b_c'=QaQ+Q_cb_cQ_c.$$

In other words, such a sum is not transferred as a whole from one legitimate system of reference to another.

Now for the product of physical quaternions. Begin with the simplest case of two factors. Leave aside ab which is split in the act of transformation, thus

$$a'b'=QaQ^2bQ,$$

and pass straight on to *the product of antivariant factors*, say,

$$H=a_cb. \tag{30}$$

Pass from the system S to any other legitimate system S'. Then $H'=Q_ca_cQ_c\,.\,QbQ$, whence, by the associative property, and remembering that $Q_cQ=1$,

$$H'=Q_cHQ. \tag{31}$$

Thus, the new quaternion H, though it is neither covariant with the standard q nor covariant with q_c, *is transformed as a whole* (composed of constituents already admitted) and can therefore be used for relativistic purposes. A moment's reflection will convince the reader that such a procedure will not infringe against the principle of relativity. And the meaning of these abstract remarks will

* Any two quaternions of the set

$$a,\ b,\ d,\ \ldots,$$

or any two of the set

$$a_c,\ b_c,\ d_c,\ \ldots,$$

being *covariant* with one another, we may conveniently call any quaternion of the first set *antivariant* with respect to any one of the second set.

become plainer when we come, in the next chapter, to consider a concrete law involving a magnitude which, in passing from S to S', is transformed exactly as the above quaternion H. Meanwhile, let us look for some further properties of that quaternion.

Consider H_c, the conjugate of H. This will be, by the elementary rule of the conjugate of a product, Quat. 8,

$$H_c = b_c a.$$

Now, transforming this, we get $H_c' = Q_c b_c Q_c Q a Q$, or, in exactly the same way as above,*

$$H_c' = Q_c H_c Q. \tag{32}$$

Thus we see that

$$Q_c[\quad]Q$$

is the relativistic transformer of *both H and its conjugate H_c*, and hence also of their sum and of their difference, *i.e.* also of the scalar and of the vector parts of the quaternion H separately, say s and $\mathbf{L}$,

$$s = \mathrm{S}H, \quad \mathbf{L} = \mathrm{V}H.$$

Now, s being a scalar, we have simply

$$s' = Q_c s Q = s Q_c Q = s,$$

i.e. s is an invariant, as was proved before. Thus, the scalar part of $a_c b$ need not detain us any further.

What we really need for the subsequent physical application is $\mathbf{L}$, the vector part of this quaternion. This is transformed into

$$\mathbf{L}' = Q_c \mathbf{L} Q, \tag{33}$$

and since Q, Q_c are unit quaternions, the tensor of $\mathbf{L}$ is an *invariant*, $\mathrm{T}\mathbf{L}' = \mathrm{T}\mathbf{L}$, which may also be written, more conveniently,†

$$\mathbf{L}'^2 = \mathbf{L}^2. \tag{34}$$

* Here, H_c' is an abbreviation for $(H_c)'$, the transformed conjugate. But taking the conjugate of the transformed quaternion, (31), we obtain at once $(H')_c = Q_c H_c Q$, so that $(H_c)' = (H')_c$, and both sides may, therefore, be written simply H_c'.

† Remember that the square of the tensor, or the norm of any quaternion X is XX_c. Now, in our case, $\mathbf{L}$ being a *scalarless* quaternion, its conjugate is $\mathbf{L}_c = -\mathbf{L}$, so that its norm is simply $-\mathbf{L}^2$. If $\mathbf{L}$ were an ordinary, real vector, we could write (instead of $-\mathbf{L}^2$) L^2, the square of its size or absolute value. But since $\mathbf{L}$ is a complex vector, or a *bivector*, the above notation is preferable. $\mathbf{L}^2$ is a *scalar*, of course, *e.g.* a complex scalar, as will be seen presently. We need not put the prefix S before it, since $\mathrm{V}\mathbf{L}\mathbf{L}$ is always *nil*, by the elementary definition of vector product.

These being the transformational properties of the vector $\mathbf{L}=\mathrm{V}a_c b$, let us see what is its structure.

Since both a and b have the structure of q, the standard of physical quaternions, write

$$a=\iota\alpha+\mathbf{A}\,; \quad \therefore \; a_c=\iota\alpha-\mathbf{A}$$

and

$$b=\iota\beta+\mathbf{B},$$

where α, β are real scalars and $\mathbf{A}$, $\mathbf{B}$ ordinary, *i.e.* real vectors. Then

$$\mathbf{L}=\mathbf{L}_1-\iota\mathbf{L}_2, \tag{35}$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are the real vectors

$$\mathbf{L}_1=\mathrm{V}\mathbf{BA}, \quad \mathbf{L}_2=\beta\mathbf{A}-\alpha\mathbf{B}. \tag{36}$$

Thus, $\mathbf{L}$ is a complex vector or a **bivector**,—called so, since it consists of two ordinary vectors. We had, in Chap. II., a sample of such a magnitude in the electromagnetic bivector. The complex invariant, (34), of $\mathbf{L}$ splits into its *two real invariants*,

$$L_1{}^2-L_2{}^2 \quad \text{and} \quad (\mathbf{L}_1\mathbf{L}_2). \tag{37}$$

The second of these invariants vanishes, since, by (36), $\mathbf{L}_1$ is perpendicular to $\mathbf{L}_2$. This being the case, $\mathbf{L}=\mathrm{V}a_c b$ is a *special* bivector (and is equivalent to Sommerfeld's 'special six-vector'). In order to obtain the *general* bivector, whose two real vectors are mutually independent, we have only to add to the above $\mathbf{L}$ another, appropriate, special bivector having the same transformational properties. For this purpose we can take the special bivector $\mathbf{L}^{(s)}$, the **supplement** of $\mathbf{L}$, defined by $\mathbf{L}^{(s)}=\mathrm{V}a_c^{(s)}b^{(s)}$, where $a^{(s)}$, $b^{(s)}$ is a pair of physical quaternions, such that

$$\mathrm{S}a^{(s)}a_c=\mathrm{S}a^{(s)}b_c=\mathrm{S}b^{(s)}a_c=\mathrm{S}b^{(s)}b_c=0.$$

But particulars concerning the choice of a sufficiently general supplement, as this is, need not detain us here.

Henceforth we shall denote by $\mathbf{L}$ the general bivector, thus obtainable. And we shall call it, where it will be needed for the sake of distinction, a *left-handed* bivector, owing to the position of the subscript $_c$ in its transforming operator, or in the generating quaternionic factors: a_c, b; $a_c^{(s)}$, $b^{(s)}$.

Similarly, starting from ab_c (where a, b are not necessarily the same as above), and proceeding as before, we can construct a

general *right-handed* bivector, $\mathbf{R}$, consisting of two ordinary, real vectors $\mathbf{R}_1$, $\mathbf{R}_2$. This will be transformed by $Q[\quad]Q_c$, *i.e.* so that

$$\mathbf{R}' = Q\mathbf{R}Q_c, \tag{38}$$

and will, therefore, have again the two real invariants

$$R_1^2 - R_2^2 \quad \text{and} \quad (\mathbf{R}_1\mathbf{R}_2). \tag{39}$$

Both $\mathbf{L}$ and $\mathbf{R}$ can be used, with equal convenience, for relativistic purposes, and will be found useful for the treatment of electromagnetic questions.

To illustrate the above properties by a simple kinematical example, take, as the generating factors, the velocity- and the acceleration-quaternions of a particle. Then

$$\mathbf{L} = \mathrm{V}\, Y_c Z = -\mathrm{V}\dot{\mathbf{r}}\ddot{\mathbf{r}} + \iota c(\dot{t}\ddot{\mathbf{r}} - \ddot{t}\dot{\mathbf{r}}),$$

i.e., after a slight calculation, in terms of the ordinary velocity $\mathbf{p}$ and acceleration $\mathbf{a}$,

$$\mathbf{L}_1 = \gamma^3 \mathrm{V}\mathbf{a}\mathbf{p}, \quad \mathbf{L}_2 = -c\gamma^3\mathbf{a}.$$

Thus, besides $(\mathbf{L}_1\mathbf{L}_2)$ which vanishes, obviously, we have the invariant ($L_1^2 - L_2^2$ and, therefore, also)

$$\frac{1}{c}\sqrt{L_2^2 - L_1^2} = a\gamma^3\sqrt{1 - \beta^2 \sin^2(\mathbf{p}, \mathbf{a})},$$

and this invariant has a simple kinematical meaning. For it is identical with the absolute value of the rest-acceleration a' of the particle, as given by (14).

Returning to our general topic, let us consider the product of any number of left-handed bivectors. Then we shall see, by (33), that, in transforming it, all the internal Q's and Q_c's, as it were, neutralize one another ($QQ_c = 1$), and what is left is only the Q_c at the beginning and the Q at the end of the whole chain, exactly as for a single $\mathbf{L}$. In other words, the vector part of the product of any number of left-handed bivectors is again a left-handed bivector. Similarly, we see, by (38), that the vector part of the product of right-handed bivectors is again a right-handed bivector. But we shall hardly find a physical application of such products.

What will turn out to be rather important for such application is the product of one of the original physical quaternions into a bivector. Of such a nature will be the ponderomotive force in an electromagnetic field.

Notice, therefore, that if a be any physical quaternion covariant with q (not necessarily that already involved in **L** or **R**), the product $a\mathbf{L}$ will transform into

$$a'\mathbf{L}' = QaQQ_c\mathbf{L}Q = Qa\mathbf{L}Q,$$

that is to say, $a\mathbf{L}$ *will again be covariant with* q. So also will $\mathbf{R}a$ be covariant with q. And similarly will $\mathbf{L}a_c$ and $a_c\mathbf{R}$ be covariant with q_c. In short symbols,

$$a\mathbf{L} \sim \mathbf{R}a \sim q, \tag{40}$$

$$\mathbf{L}a_c \sim a_c\mathbf{R} \sim q_c. \tag{40a}$$

Each of these products can be, and is, in fact, used for relativistic purposes. As regards their structure, they are **biquaternions**, in Hamilton's (not in Clifford's) sense of the word, that is to say, quaternions, of which both the scalar and the vector parts are complex.* But, as we shall see in the next chapter, any one of such biquaternions can be split into a pair of our original physical quaternions, each $\simeq q$ or $\simeq q_c$ in the case of (40) or (40*a*) respectively. In this way we fall back to the quaternions considered at the outset.

Thus, the product of *any* number of *antivariant* physical quaternions

$$\dots ab_c de_c \dots$$

will furnish us (after the rejection of the invariant scalar part) a bivector **L** or **R**, which is transformed by $Q_c[\quad]Q$ and $Q[\quad]Q_c$ respectively, or again, biquaternions consisting of pairs of primary physical quaternions, which are transformed by $Q[\quad]Q$, or by $Q_c[\quad]Q_c$.

And, as was already remarked, products of covariant factors, such as ab, are out of question.

As concerns the operation of division by a physical quaternion, we know that it is reduced to multiplication by its reciprocal. Thus, it will be enough to observe that the *reciprocal* of a physical quaternion is again a physical quaternion. For we have

$$a^{-1} = a_c(\mathrm{T}a)^{-2},$$

* Thus, for example, if $a = \iota\alpha + \mathbf{A}$, and $\mathbf{L} = \mathbf{L}_1 - \iota\mathbf{L}_2$,

$$\mathrm{S}a\mathbf{L} = -(\mathbf{AL}_1) + \iota(\mathbf{AL}_2), \quad \mathrm{V}a\mathbf{L} = \alpha\mathbf{L}_2 + \mathrm{V}\mathbf{AL}_1 + \iota\alpha\mathbf{L}_1 - \iota\mathrm{V}\mathbf{AL}_2.$$

and the tensor $\mathrm{T}a$ is a relativistic invariant. Notice that a and a^{-1} are mutually antivariant.

Finally, notice that any one of the above factors may be replaced by the quaternionic differential operator

$$D = \frac{\partial}{\partial l} + \nabla \simeq q,$$

or by its conjugate D_c, which is $\simeq q_c$. Thus, for example, if the quaternion $\Phi \simeq q$ be a function of time and the coordinates, then $\mathrm{V}D_c\Phi$ will be a left-handed bivector; and so also will $\mathrm{V}D\Phi_c$ be a right-handed bivector. For, independently of their differentiating power, these operators behave with respect to the Lorentz transformation exactly as any of our primary quaternionic magnitudes.

CHAPTER VIII.

FUNDAMENTAL ELECTROMAGNETIC EQUATIONS.

In this chapter we shall consider, from the relativistic standpoint, the fundamental, or 'microscopic,' equations of the electron theory and their consequences. These equations, written in their ordinary vector form, are, as under (I.) and (II.), Chapter II.,

$$\left.\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} + \rho\mathbf{p} &= c\,.\,\mathrm{curl}\,\mathbf{M}\,; \quad \rho = \mathrm{div}\,\mathbf{E} \\ \frac{\partial \mathbf{M}}{\partial t} &= -c\,.\,\mathrm{curl}\,\mathbf{E}\,; \quad \mathrm{div}\,\mathbf{M} = 0 \end{aligned}\right\} \qquad \text{(I.)}$$

and

$$\mathbf{P} = \rho\left[\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{p}\mathbf{M}\right] \equiv \rho\mathfrak{F}. \qquad \text{(II.)}$$

Here, $\mathbf{p}$ is the velocity of a charge-element with respect to that framework S, for which, to begin with, the equations are supposed to be rigorously valid; $\mathbf{P}$ is the ponderomotive force, per unit volume, and $\mathfrak{F}$ the ponderomotive force per unit charge, or *the electric force.*

First of all, we have to ask whether these equations satisfy the principle of relativity, that is to say, whether they preserve their form when we pass from the system $S(t, x, y, z)$ to another system $S'(t', x', y', z')$ moving with uniform velocity relatively to S. And if the answer be, as it is in fact, in the affirmative, what are the connexions between $\mathbf{E}'$, $\mathbf{M}'$, the dielectric displacement and the magnetic force as estimated from the S'-standpoint, and these field-vectors as estimated by the S-observers? To answer both of these questions, first with regard to the differential equations (I.), we could follow the way originally taken by Einstein, *viz.* subject the time and the coordinates involved in the differential operators to the

Lorentz transformation $x=\gamma_v(x'+vt')$, etc., and expressing $\mathbf{p}$ in terms of $\mathbf{p}'$ by means of his addition theorem of velocities, show the invariance of the form of these equations, and finally gather together the terms which in the transformed equations play the part of the field-vectors.* But the shortest method to obtain these results is to write the four equations (I.) in their condensed quaternionic form,

$$D\mathbf{B}=C, \tag{1}$$

as given in Chap. II., and to test the constituents of this equation with regard to their relativistic qualities.

Here, it will be remembered, $\mathbf{B}=\mathbf{M}-\iota\mathbf{E}$, while

$$C=\rho\left[\iota+\frac{1}{c}\mathbf{p}\right], \tag{2}$$

or, in terms of the velocity-quaternion, ($3a$), Chap. VII.,

$$C=\frac{\rho}{c\gamma_p}Y, \tag{2a}$$

where $\gamma_p=(1-p^2/c^2)^{-\frac{1}{2}}$.

Keeping this in mind, consider the equation (1). We know already that the differentiator D behaves exactly as a physical quaternion, *viz.* that $D\simeq q$. The only thing, therefore, we still require, is to find the nature of the current-quaternion C.

Now, the electric charge de of any individual portion of an electron is a relativistic invariant, *i.e.* if dS be the volume of that portion, and dS' its S'-correspondent, then

$$\rho\, dS=\rho'\, dS'. \tag{3}$$

In fact, taking the divergence of the first of (I.), we have

$$0=\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{p})=\frac{\partial\rho}{\partial t}+(\mathbf{p}\nabla)\rho+\rho\operatorname{div}\mathbf{p},$$

which is known as 'the equation of continuity,' or, denoting by $\frac{d}{dt}$ the rate of individual change, as on p. 31,

$$\frac{d\rho}{dt}+\rho\operatorname{div}\mathbf{p}=0,$$

* An outline of this way of treatment, which may be helpful to some readers, will be found in **Note 1** at the end of the chapter.

whence, multiplying by dS and observing that $\frac{d}{dt}(dS) = dS.\,\mathrm{div}\,\mathbf{p}$,*

$$\frac{d}{dt}(\rho\, dS) = \frac{d}{dt}(de) = 0.$$

Thus, the charge, as estimated from the S-standpoint, is invariable in time, notwithstanding the motion and deformation of the volume-element we are watching. This being the case, we can imagine the charge first fixed in S and then set it into motion, bringing it by and by to the velocity $\mathbf{v}$, when it will be at rest in S'. Claiming, therefore, in the name of the principle of relativity, the same rights for S' as for S, we shall have $de' = de$. (If the reader does not like this kind of proof, he can simply postulate the invariance of charge, and verify *a posteriori*, after having obtained $\mathbf{E}'$ in terms of $\mathbf{E}$, $\mathbf{M}$, that this postulate is fulfilled.)

On the other hand, remembering that volumes are transformed in the same way as longitudinal dimensions, and denoting for the moment by dS_0 the rest-volume of the element considered, we shall have

$$dS = dS_0\sqrt{1 - p^2/c^2} \quad \text{and} \quad dS' = dS_0\sqrt{1 - p'^2/c^2}$$

or

$$\gamma_p\, dS = \gamma_{p'}\, dS'.$$

Therefore, by (3),

$$\frac{\rho}{\gamma_p} = \frac{\rho'}{\gamma_{p'}},$$

that is to say, ρ/γ_p, the coefficient of Y in (2a), is an invariant.

Now, as we know from the last chapter, Y is a physical quaternion. Therefore, C, the *current-quaternion*, as it was already called in Chapter II., is again a physical quaternion, like the standard q,

$$C \simeq q,$$

as well as $D \simeq q$.

This proves *the invariance of the form of the equation* (1), or of the equations (I.), with respect to the Lorentz transformation, and gives at the same time the connexion between $\mathbf{B}'$ and $\mathbf{B}$.

In fact, since $C' = QCQ$, we have from (1)

$$QD\mathbf{B}Q = C',$$

* Cf. my *Vectorial Mechanics*, p. 126.

and inserting $QQ_c = 1$ between D and $\mathbf{B}$,

$$D'Q_c\mathbf{B}Q = C',$$

i.e.

$$D'\mathbf{B}' = C', \tag{1'}$$

where $\mathbf{B}' = Q_c\mathbf{B}Q$.* Thus, $\mathbf{B}$, the electromagnetic bivector, is a left-handed bivector.

Or, to obtain this bivector in its typical form $\mathrm{V}a_c b$, we may proceed as follows. Operate on both sides of (1) with D_c. Then

$$D_c D\mathbf{B} = D_c C.$$

But $D_c D$ is an invariant. This, therefore, is already the required form. We need not even put the prefix V before $D_c C$, since $\mathrm{S}D_c C = 0$, as we shall see when we next return to the last equation.

Thus, $\mathbf{B}$ *is a left-handed bivector*, having the same structure and the same transformational properties as our $\mathbf{L}$ of the last chapter. Henceforth we can consider it as the standard of **physical bivectors**, in the same way as q has been the standard of physical quaternions.

It will be found convenient for subsequent work to write throughout $\mathbf{L}$ (instead of our previous $\mathbf{B}$) for the electromagnetic bivector,† thus:

$$\mathbf{L} = \mathbf{M} - \iota\mathbf{E}. \tag{4}$$

The quaternionic equivalent of the electromagnetic differential equations (I.) *will now be*

$$D\mathbf{L} = C, \tag{I. a}$$

and the transformation formula of the electromagnetic bivector

$$\mathbf{L}' = Q_c\mathbf{L}Q. \tag{5}$$

The invariance of the formula (II.) for the ponderomotive force will, with equal ease, be proved later on. Meanwhile let us fix our attention upon (5).

As already pointed out in the last chapter, Q and Q_c being unit quaternions, the square of the electromagnetic bivector is an invariant, *i.e.*

$$\mathbf{L}'^2 = \mathbf{L}^2.$$

* That the product $Q_c\mathbf{B}Q$ is, in fact, a pure vector (*i.e.* a *scalarless* quaternion), like $\mathbf{B}$, we see at a glance. For the conjugate of $Q_c\mathbf{B}Q$ is $Q_c\mathbf{B}_c Q = -Q_c\mathbf{B}Q$, so that the sum of that product and of its conjugate is *nil*. Q.E.D.

† And correspondingly, in what follows, $\mathbf{R}$ for the complementary bivector $\mathbf{M} + \iota\mathbf{E}$, which will turn out to be right-handed.

Now, by (4),

$$-\mathbf{L}^2 = M^2 - E^2 - 2\iota(\mathbf{EM}),$$

and similarly for $\mathbf{L}'^2$. Thus we have the two real *invariants*

$$\tfrac{1}{2}(M^2 - E^2) \quad \text{and} \quad (\mathbf{EM}). \tag{6}$$

The first of these invariants, the difference of the densities of the magnetic and the electric energies, is the electromagnetic *Lagrangian function* per unit volume.* The second invariant, the scalar product of **E** and **M**, has no particular name of its own. Notice that what is called a *pure* electromagnetic wave is defined by $M^2 = E^2$ and $(\mathbf{EM}) = 0$. In words: energy half electric and half magnetic, and **E** and **M** perpendicular to one another. Using the electromagnetic bivector we can characterize pure waves more shortly by $\mathbf{L}^2 = \mathbf{LL} = 0$. At the same time we see that a wave which is pure from the S-standpoint is equally pure from the S'-point of view. In short, purity, at least in this domain of relations, is an invariant property. But this only by the way.

Next, to develop (5) into its vectorial form, remember that, by (50), Chap. V.,

$$Q = \cos\frac{\omega}{2} + \mathbf{u} \,.\, \sin\frac{\omega}{2},$$

where $\mathbf{u}$ is the unit of $\mathbf{v}$, the velocity of S' relative to S, and where ω is the imaginary angle previously defined. Multiply out the right side of (5). Then

$$\mathbf{L}' = (1 - \cos\omega) \,.\, \mathbf{u}(\mathbf{uL}) + \cos\omega \,.\, \mathbf{L} + \sin\omega \,.\, \mathrm{V}\mathbf{Lu}.$$

From this intermediate form we can easily see that $\mathbf{L}'$ is obtained from **L** *by turning it about* **u**, *the axis of the quaternion* Q, *through* ω, *the double of the angle of that quaternion.* Such then is the office of the operator $Q_c[\quad]Q$. This is only a particular instance of a theorem of the calculus of quaternions, given by Hamilton himself.†

* The properties of this function, belonging to the elements of the Electron Theory, are given in **Note 2.**

† If k be any quaternion, k^{-1} its reciprocal, and x any quaternion to be operated on, then the operator $k^{-1}[\quad]k$ turns the vector of x about the axis of k through double the angle of k, while the scalar (s) of x remains unchanged, of course (since $k^{-1}sk = sk^{-1}k = s$). Cf. Tait's *Quaternions*, 1890, p. 75.

But let us write the last formula in terms of γ, which is an abbreviation for $\gamma_v = (1-\beta^2)^{-\frac{1}{2}}$, $\beta = v/c$. Remembering that $\cos\omega = \gamma$ and $\sin\omega = \iota\beta\gamma$, we have

$$\mathbf{L}' = (1-\gamma)\mathbf{u}(\mathbf{uL}) + \gamma\mathbf{L} + \frac{\iota}{c}\gamma\mathrm{V}\mathbf{Lv},$$

or, employing again the longitudinal stretcher ϵ, of ratio γ,

$$\mathbf{L}' = \gamma\left[\frac{1}{\epsilon}\mathbf{L} + \frac{\iota}{c}\mathrm{V}\mathbf{Lv}\right], \tag{7}$$

and splitting into the real and the imaginary parts, according to (4),

$$\left.\begin{aligned} \mathbf{E}' &= \gamma\left[\frac{1}{\epsilon}\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{vM}\right] \\ \mathbf{M}' &= \gamma\left[\frac{1}{\epsilon}\mathbf{M} - \frac{1}{c}\mathrm{V}\mathbf{vE}\right]. \end{aligned}\right\} \tag{7a}$$

Or, finally, in Cartesian expansion, using $_{1,2,3}$ for the rectangular components of the vectors taken along the direction of motion and perpendicular to it (right-handed system of axes),

$$\left.\begin{aligned} E_1' &= E_1, \quad E_2' = \gamma(E_2 - \beta M_3), \quad E_3' = \gamma(E_3 + \beta M_2) \\ M_1' &= M_1, \quad M_2' = \gamma(M_2 + \beta E_3), \quad M_3' = \gamma(M_3 - \beta E_2). \end{aligned}\right\} \tag{7b}$$

These are the relativistic formulae for the transformation of the electric and the magnetic vectors, as obtained by Einstein. They agree entirely with those given by Lorentz in his modified theory (see p. 86). Notice that, in passing from the S- to the S'-standpoint, the longitudinal components of **E**, **M** remain unchanged, while the changes brought about in their transversal components involve the vector products V**vM** and V**Ev** and the coefficient γ.

Multiplying both sides of (5) by Q as a prefactor and by Q_c as a postfactor, we have at once

$$\mathbf{L} = Q\mathbf{L}'Q_c. \tag{5'}$$

But Q_c follows from Q, and *vice versa*, by a mere change of the sign of **v**. Thus, the inverse transformation, giving **E**, **M** in terms of **E**′, **M**′, is obtained by changing the sign of **v** in the vector formulae, or by writing $-\beta$ instead of β in their Cartesian expansions, and by transferring the dashes, to wit

$$E_2 = \gamma(E_2' + \beta M_3'), \quad E_3 = \gamma(E_3' - \beta M_2'), \text{ etc.,}$$

as the reader may also prove by solving (7*b*). This shows once more that none of the systems of reference is privileged.

The invariance of electric charge, used at the outset, can now be directly verified by differentiation of the transformed electric vector or of its components.*

The applicability of the above formulae of transformation is obvious. For, if we know a solution of the electromagnetic differential equations for one of the legitimate systems of reference, we can deduce from it at once the solution for any other of such systems. Now, the problem of integration may be much easier for one of these systems than for any other, owing to some particular simplicity of the conditions as stated from the standpoint of the former system. Whence the advantage of the method.†

The simplest solution of the electromagnetic equations is an *electrostatic* field corresponding to a given distribution of charges (electrons), which are all fixed with respect to a legitimate framework, say S'. The S-correspondent of this will be the electromagnetic field accompanying a system of electrons *in uniform translational motion*, with velocity $\mathbf{v}$ relative to S, or what is called a **convective field.** The framework S' will be the rest-system belonging permanently to these charges. It will be good, before proceeding further with our general subject, to consider this example at some length.

Let us suppose, therefore, that we have in S' a purely electrostatic field, so that $\mathbf{E}' = -\nabla'\phi'$, where ϕ' is the scalar potential of the given distribution of charge, while $\mathbf{M}' = 0$. Then, remembering that the inverse of the first of (7*a*) is

$$\mathbf{E} = \gamma\left[\frac{1}{\epsilon}\mathbf{E}' - \frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{M}'\right],$$

we shall have, from the S-point of view,

$$\mathbf{E} = \gamma\epsilon^{-1}\mathbf{E}',$$

i.e., in Cartesians,

$$E_1 = E_1', \quad E_2 = \gamma E_2', \quad E_3 = \gamma E_3'.$$

* See **Note 3.**

† The reader will find it useful to compare this procedure carefully with that contained in Lorentz's 'Theorem of corresponding states,' as given in Chapter III.

The second of (7*a*) gives us at once **M** in terms of **E**, *viz.*

$$\mathbf{M} = \frac{\epsilon}{c} \mathrm{V}\mathbf{vE} = \frac{1}{c} \mathrm{V}\mathbf{vE},$$

since the stretcher ϵ acts along **v**, while the vector product is normal to **v**.

Thus we have for the most general *convective field*, accompanying any system of charges which moves as a whole with the uniform translational velocity **v** relative to S,

$$\left.\begin{aligned} \mathbf{E} &= \gamma\epsilon^{-1}\mathbf{E}' \\ \mathbf{M} &= \frac{1}{c} \mathrm{V}\mathbf{vE}. \end{aligned}\right\} \qquad (8)$$

Here $\mathbf{E}' = -\nabla'\phi'$, the scalar function ϕ' being the electrostatic potential of the given distribution of charge fixed in S'. The problem is therefore reduced to finding, for each particular case of distribution, the scalar potential ϕ'. Observe that this is the potential of $\mathbf{E}'$, while **E** has no such potential. Notice, further, that the magnetic lines, due to the motion of charges, are everywhere normal to both **E** and the direction of motion. And since $\mathbf{E}'$ is coplanar with **E**, **v**, the magnetic lines are also at right angles to $\mathbf{E}'$.

The gradient or slope $\nabla'\phi'$ can easily be replaced by $\nabla\phi'$. In fact, measuring x along the direction of motion, so that $x' = \gamma(x - vt)$, and remembering that, by assumption, $\partial\phi'/\partial t' = 0$, we have

$$\frac{\partial\phi'}{\partial x} = \gamma\frac{\partial\phi'}{\partial x'}, \quad \frac{\partial\phi'}{\partial y} = \frac{\partial\phi'}{\partial y'}, \quad \frac{\partial\phi'}{\partial z} = \frac{\partial\phi'}{\partial z'},$$

i.e.

$$\epsilon\nabla'\phi' = \nabla\phi',$$

so that the first of (8) can be written

$$\mathbf{E} = -\gamma\epsilon^{-2}\nabla\phi'.$$

Thus, the displacement **E**, as already remarked, has no scalar potential. But the *electric force* $\mathfrak{F}$, or the ponderomotive force per unit of charge carried along with S', has such a potential, exactly as in Lorentz's treatment, given in Chapter III. p. 81. In fact, remembering that in the present case $\mathbf{p} = \mathbf{v}$, we have by (II.) and by the second of (8),

$$\mathfrak{F} = \mathbf{E} + \frac{1}{c^2} \mathrm{V}\mathbf{v}\mathrm{V}\mathbf{vE} = \mathbf{E} - \beta^2[\mathbf{E} - \mathbf{u}(\mathbf{uE})],$$

or

$$\mathfrak{F} = \gamma^{-2}\epsilon^2 \mathbf{E},$$

and by our last formula,

$$\mathfrak{F} = -\nabla\left(\frac{\phi'}{\gamma}\right). \tag{9}$$

Thus ϕ'/γ is the scalar potential of the electric force. This is the *convection potential* of Chap. III., the above equation being identical with formula (21) of that chapter, in which ϕ was $\gamma\phi'$. The same result may be deduced more directly from the transformational properties of the ponderomotive force, to be developed later on.

Since γ is constant throughout S', the surfaces of constant convection potential and those of constant ϕ' overlap. We see, therefore, that the lines of electric force $\mathfrak{F}$ (but not those of displacement $\mathbf{E}$) cut *perpendicularly* the surfaces of constant electrostatic potential of the rest-system, $\phi' = \text{const}$. The electric force and displacement of that system are identical, of course, *i.e.* $\mathfrak{F}' = \mathbf{E}'$.

To illustrate the general formulae (8) of the convective field, suppose that the distribution of electric charge in S' is in homogeneous concentric *spherical* sheets round O', the origin of the coordinates or the origin of the vectors $\mathbf{r}'$. Then ϕ', and consequently also E', will be functions of r' alone, and the lines of displacement in S' will be straight and radial or, say,

$$\mathbf{E}' = f(r') \,.\, \mathbf{r}', \tag{10'}$$

where f is a scalar function of its argument. By the fundamental formulae of transformation, $\mathbf{r}' = \epsilon\mathbf{r} - \mathbf{v}\gamma t$. Now, since the whole field, together with the charges, moves past S without being deformed, it is enough to consider it at one single instant. Let this be the instant $t = 0$, when O' coincides with O, the origin of the S-coordinates or of all vectors $\mathbf{r}$. Then

$$\mathbf{r}' = \epsilon\mathbf{r},$$

and, by (8),

$$\left.\begin{aligned} \mathbf{E} &= \gamma f(r') \,.\, \mathbf{r} \\ \mathbf{M} &= \frac{1}{c}\gamma f(r') \,.\, \mathrm{V}\mathbf{v}\mathbf{r}, \end{aligned}\right\} \tag{10}$$

so that the dielectric displacement in S is again in straight radial lines, while the magnetic lines are circles normal to the direction of motion and centred upon the axis of symmetry passing through O.

The whole electromagnetic field is symmetrical, of course, round this longitudinal axis. Since $\mathbf{r}=\epsilon^{-1}\mathbf{r}'$, or

$$x=\frac{x'}{\gamma}=x'\sqrt{1-\beta^2}, \quad y=y', \quad z=z',$$

the spheres $r'=const.$ become, in S, oblate ellipsoids of revolution, as in the FitzGerald-Lorentz contraction, *i.e.* having

$$\frac{r'}{\gamma}, \ r', \ r'$$

for their semiaxes. These are known as *Heaviside ellipsoids.* Such then will be the surfaces of constant convection potential, and the lines of electric force ($\mathfrak{F}$), cutting these ellipsoids at right angles, will be parabolic arcs, contained in the meridian planes.

If $s=(y^2+z^2)^{\frac{1}{2}}$ be the distance of a point from the axis of symmetry, we have

$$r'=\sqrt{\gamma^2 x^2+s^2},$$

or also, denoting by θ the angle contained between $\mathbf{r}$ and the axis,

$$r'=\gamma r\sqrt{1-\beta^2 \sin^2\theta}. \qquad (11)$$

This is to be substituted in each particular case for the argument of the given function f in (10).

Take, as the simplest case of the above kind, a single sphere of homogeneous surface-charge, or a Lorentz electron. Call its rest-radius R and its total charge e (which, as we know, is the same thing as e'). Then $E'=0$ inside the sphere $r'=R$, and consequently also $E=0$ inside the oblate ellipsoid $\gamma^2x^2+s^2=R^2$, while at the surface of and outside the electron*

$$\mathbf{E}'=\frac{e\mathbf{r}'}{4\pi r'^3},$$

and therefore

$$\mathbf{E}=\frac{e\gamma}{4\pi r'^3}\mathbf{r}, \quad r'\geqq R,$$

that is, by (11),

$$\mathbf{E}=\frac{e}{4\pi r^2}\cdot\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\frac{\mathbf{r}}{r}, \qquad (12)$$

with the magnetic force $\mathbf{M}=\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}$ to match. For any given θ the value of E, and consequently also that of M, are inversely propor-

* In Heaviside's rational units.

tional to the square of the distance from the centre of the electron. The unit tubes of displacement, though everywhere radial, are crowded towards the equator, and the more so, the greater the velocity of motion. At any given distance r, the density of the tubes at the equator is greater than that at the poles ($\theta = 0$ or π) in the ratio $E_{\pi/2} : E_0 = 1 : (1-\beta^2)^{\frac{3}{2}}$.

From the above, widely known, formulae the longitudinal and the transversal electromagnetic masses of the electron may be easily deduced in the usual way. The flux of energy or the Poynting vector being

$$\mathfrak{P} = c\mathrm{V}\mathbf{EM} = \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{vE} = E^2\mathbf{v} - (\mathbf{Ev})\mathbf{E},$$

we have for the electromagnetic momentum, per unit volume, by (30), Chap. II.,

$$\mathbf{g} = \frac{v}{c^2}[E^2\mathbf{u} - E_1\mathbf{E}],$$

where $\mathbf{u}$ is the unit of $\mathbf{v}$ and E_1 the longitudinal component of $\mathbf{E}$. Integrating through the whole field (from $r' = R$ till $r' = \infty$) and taking advantage of its axial symmetry, we obtain, for the total *electromagnetic momentum*,*

$$\mathbf{G} = \frac{e^2}{6\pi c^2 R}\gamma\mathbf{v}, \tag{13}$$

whence the *longitudinal electromagnetic mass* m_l of the electron and the *transversal* one, m_t, defined by $m_l = dG/dv$, $m_t = G/v$:

$$m_l = m_0\gamma^3, \quad m_t = m_0\gamma, \tag{14}$$

where

$$m_0 = \frac{e^2}{6\pi c^2 R}. \tag{15 surf.}$$

These are the well-known formulae of Lorentz, as mentioned previously. They are valid for an electron of homogeneous surface-charge. In the case of volume-charge, we should obtain for the electromagnetic momentum $\frac{6}{5}$ of the above value, so that (14) would continue to hold with m_0 equal to $\frac{6}{5}$ of the above,

$$m_0 = \frac{e^2}{5\pi c^2 R}. \tag{15 vol.}$$

* See **Note 4** at the end of the chapter.

The electromagnetic momentum can, in either case, be written

$$\mathbf{G} = m_0 \gamma \mathbf{v}. \tag{16}$$

Thus, m_0, the electromagnetic rest-mass, plays the same part as the rest-mass, of any origin, in the relativistic dynamics of a particle. Cf. (24), Chap. VII.

Having for the present sufficiently illustrated the transformational properties of the electromagnetic bivector, let us now return to our general subject.

Consider again the equation

$$D\mathbf{L} = C \tag{1.a}$$

embodying in itself the whole of the electronic differential equations (I.), and showing at the same time their invariance. Operate upon both of its sides with D_c. Then

$$D_c D\mathbf{L} = D_c C.$$

But $D_c D$ is the *Dalembertian*,

$$D_c D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \square,$$

and this is a purely scalar operator; that is to say, if applied to a scalar it gives a scalar, and if applied to a vector it gives again a vector. Now, $\mathbf{L}$ is scalarless. Therefore

$$\mathrm{S}D_c C = 0. \tag{17}$$

This is *the equation of continuity*. In fact, its developed form is, by (2),

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{p}) = 0.$$

But this only by the way.

Next, introduce an auxiliary quaternion Φ, satisfying the differential equation

$$\square \Phi = -C \tag{18}$$

and the supplementary condition

$$\mathrm{S}D_c \Phi = 0. \tag{19}$$

Then, when Φ is found, for any prescribed C, the electromagnetic bivector will be given by

$$\mathbf{L} = -D_c \Phi. \tag{20}$$

Now, $D_c D = \Box$, being the norm of $D \simeq q$, will be an invariant, as was already remarked on p. 113. Therefore, by (18), Φ will be a physical quaternion, having an imaginary scalar and a real vector. Write it, therefore,

$$\Phi = \iota\phi + \mathbf{A} \simeq q, \tag{21}$$

and call it the **potential-quaternion**, since the whole electromagnetic bivector is derived from it by simple differentiation. The corresponding world-vector is called the *four-potential.*

The scalar part of Φ is ι times the usual *scalar potential*, and its vector part is the *vector potential.* In fact, splitting (20) into the real and the imaginary parts, we obtain at once

$$\mathbf{M} = \mathrm{V}\nabla\mathbf{A} = \operatorname{curl} \mathbf{A},$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},$$

while the condition (19) becomes

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \operatorname{div} \mathbf{A} = 0,$$

and these are the familiar formulae of the electron theory, as employed incidentally in Chapter III., p. 80. The differential equation (18) splits, of course, into the familiar pair of equations,

$$\Box\phi = -\rho\,; \quad \Box\mathbf{A} = -\frac{1}{c}\rho\mathbf{p},$$

identical with (16), Chap. III.

According to (21), ϕ and $\mathbf{A}$ are transformed as ct and $\mathbf{r}$. Thus, for instance, if we have in S' a purely electrostatic field, *i.e.* if $A' = 0$, then, for the convective field, as estimated from the S-standpoint,

$$\phi = \gamma\phi', \quad \gamma = \gamma_v,$$

as mentioned above, and

$$\mathbf{A} = \frac{1}{c}\mathbf{v}\gamma\phi' = \frac{1}{c}\mathbf{v}\phi,$$

as in (19), Chap. III.

So much as regards the potential-quaternion and its relationship to the electromagnetic bivector.

Next, observe that instead of the above $\mathbf{L} = \mathbf{M} - \iota\mathbf{E}$ we might equally well have taken the complex vector

$$\mathbf{R} = \mathbf{M} + \iota\mathbf{E}, \tag{22}$$

which can be called the *complementary* electromagnetic bivector. Then we would have obtained as the condensed equivalent of the fundamental equations (I.), instead of and in exactly the same way as (I. a),

$$D_c\mathbf{R} = C_c, \tag{I.b}$$

where C_c is the conjugate current-quaternion $\rho(\iota - \mathbf{p}/\mathbf{c})$. Operate on both sides of this equation with D. Then the result will be $\Box\mathbf{R} = DC_c$. And since the Dalembertian is an invariant, we see at once that $\mathbf{R}$ is a *right-handed* physical bivector,* *i.e.* that

$$\mathbf{R}' = Q\mathbf{R}Q_c. \tag{23}$$

Henceforth $\mathbf{R}$ can be considered as the standard of all such bivectors, just as $\mathbf{L}$ became the standard of the left-handed ones. Obviously, the differential equation (I.b) is invariant with respect to the Lorentz transformation, *i.e.*

$$D_c{}'\mathbf{R}' = C_c'.$$

(I.a) and (I.b) differ, of course, only formally from one another; each, when split, gives the four electromagnetic differential equations (I.). Thus, as far as the equations of the field and all their consequences are concerned, we do not need both $\mathbf{L}$ and $\mathbf{R}$, but require only one of them at a time.

For some other purposes, however, the simultaneous use of both bivectors will prove to be very advantageous.

Their symbols, being the initials of 'left' and 'right,' are chosen so as to remind the reader of their transformational properties. In connexion with these, $\mathbf{L}$ can admit a physical quaternion, covariant with q, only on its left as neighbour, and $\mathbf{R}$ only on its right. And *vice versa*, if the neighbour is covariant with q_c.

Now for the outstanding proof of the invariance of the fundamental formula (II.) for *the ponderomotive force*. To obtain this proof we have only to write that formula in terms of legitimate relativistic magnitudes.

If we multiply our left-handed electromagnetic bivector, on the left side, by any physical quaternion $\sim q$, then, as in (40), Chap. VII.,

* This property of $\mathbf{R} = \mathbf{M} + \iota\mathbf{E}$ may also be deduced directly from that of $\mathbf{L} = \mathbf{M} - \iota\mathbf{E}$. For it is easily proved that if (for any pair of real vectors $\mathbf{A}$, $\mathbf{B}$)

$\mathbf{A} - \iota\mathbf{B}$ is a *left-handed* physical bivector,

then $\mathbf{A} + \iota\mathbf{B}$ is a *right-handed* physical bivector,

and *vice versa*. (See **Note 5.**) This simple theorem will be found useful later on.

the resulting product will again be transformed like q. Now, the current-quaternion C being precisely such a quaternion, consider the full product

$$C\mathbf{L}.$$

This then will again be transformed by $Q[\quad]Q$. Develop it, by (2) and (4). Then the result will be

$$C\mathbf{L} = F + \iota F_m, \tag{24}$$

where

$$F = \rho\left[\frac{\iota}{c}(\mathbf{pE}) + \mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{pM}\right], \tag{24e}$$

and F_m, the magnetic analogue* of this,

$$F_m = \rho\left[\frac{\iota}{c}(\mathbf{pM}) + \mathbf{M} - \frac{1}{c}\mathrm{V}\,\mathbf{pE}\right]. \tag{24m}$$

Now, the vector part of F is exactly $\mathbf{P}$, the ponderomotive force per unit volume, as given by (II.), and the scalar part of F is ι/c times the activity of this force. Thus,

$$F = \frac{\iota}{c}(\mathbf{Pp}) + \mathbf{P}. \tag{25}$$

Observe that the whole product $C\mathbf{L}$, though covariant with the standard q, has not the structure of q, since it is a full biquaternion, in the Hamiltonian sense of the word. But F, and its magnetic analogue, have each the structure of q, *i.e.* a real vector and an imaginary scalar.

Similarly, the complementary $\mathbf{R}$ being a right-handed bivector, multiply it on the right side by C. Then the product $\mathbf{R}C$ will again be transformed by $Q[\quad]Q$. Develop it. Then, by (2) and (22),

$$\mathbf{R}C = -F + \iota F_m, \tag{24a}$$

with precisely the same meanings of F and F_m as above. This again is a full biquaternion.

Now, since both biquaternions, $C\mathbf{L}$ and $\mathbf{R}C$ are transformed by $Q[\quad]Q$, this will also be the relativistic transformer of their sum and of their difference. Leave alone the sum, which would give the

* The reader will have remarked that in this and in all other cases the magnetic analogue is obtained from the electric original, and *vice versa*, by writing $\mathbf{M}$ for $\mathbf{E}$, and $-\mathbf{E}$ for $\mathbf{M}$. In the present case, F_m has, as far as we know, no immediate physical meaning. And since we shall need F only, it seemed convenient to leave it without the subscript e.

physically uninteresting F_m, and take half the difference of (24) and (24*a*). This will give

$$F = \tfrac{1}{2}[C\mathbf{L} - \mathbf{R}C]. \qquad (\text{II}.a)$$

Thus, we see that F taken by itself (as well as F_m) is covariant with q. And since F has also the structure of q, it is a physical quaternion, and may as such be called the **force-quaternion per unit volume.** It has a dynamic vector, the ponderomotive force per unit volume, and an energetic scalar, proportional to the activity of that force.

At the same time we have obtained for F the expression (II.*a*), and we know that the vector part of this is equal to $\mathbf{P}$ as given by (II.). Now, (II.*a*) transforms into

$$F' = QFQ = \tfrac{1}{2}[C'\mathbf{L}' - \mathbf{R}'C'],$$

and the vector part of this quaternion is again

$$\mathbf{P}' = \rho'[\mathbf{E}' + \frac{1}{c}\mathrm{V}\,\mathbf{p}'\mathbf{M}'],$$

which proves explicitly the invariance of the formula (II.) with respect to the Lorentz transformation.

Thus, the whole of 'the fundamental equations for the vacuum,' as (I.) and (II.) are called, satisfy rigorously the principle of relativity, and it was for this reason possible to incorporate them entirely in the new doctrine.

By (25) we have, identically,

$$\mathrm{S}FC_c = 0, \qquad (26)$$

and therefore also, by (2*a*),

$$\mathrm{S}FY_c = 0. \qquad (27)$$

In four-dimensional language we should say that the four-force, equivalent to the quaternion F, is *perpendicular* to the four-current, and consequently also to the world-line of the element of electric charge acted upon. We met with this property when treating the dynamics of a particle moving under the action of a force of any nature whatever. See (21), Chapter VII.

Remember that what is, in our present case of electromagnetic action, a physical quaternion is the force-quaternion F per unit volume. That is to say, what is transformed as $\mathbf{r}$ (and ct) is $\mathbf{P}$, the ponderomotive force *per unit volume* (and c^{-1} times its activity), and not the total force acting upon an electron or upon its volume-element.

The latter is not the vector part of a physical quaternion. But, on the other hand, we know that

$$\gamma_p \times \text{volume}$$

is an invariant. Therefore

$$\gamma_p \times \text{vol.} \times F \simeq q,$$

that is to say, γ_p *times the force-quaternion calculated for any particle of electricity is again a physical quaternion.* Such then is the transformational property of ponderomotive forces due to an electromagnetic field.

Now, if one of these forces is in equilibrium with a force of any other origin, from the standpoint of the system S' (so that the particle acted upon is at rest with respect to that system), then these two forces have also to balance each other when estimated from the standpoint of any other legitimate system S. For relatively to S, the particle in question will move uniformly. Hence the requirement, that *ponderomotive forces of any origin shall be transformed in exactly the same way as those of electromagnetic origin*,* *i.e.* so that

$$\gamma_p[\text{total force} + \frac{\iota}{c} \text{ times its activity}] \simeq \mathbf{r} + \iota ct.$$

Here 'total force' means the force acting upon a particle whose velocity relative to S is $\mathbf{p}$, or upon a body of any dimensions if all its parts happen to have the same velocity.

Now, what in Chap. VII. has been called the Newtonian force, $\mathbf{N}$, satisfies exactly this relativistic requirement. In fact, according to the formula (18) of that chapter (where γ stands for γ_p),

$$\gamma_p(\mathbf{N} + \iota\nu) = X$$

is a physical quaternion, and, as we have seen, $\nu = \frac{1}{c}(\mathbf{Np})$. It is precisely for this reason that the Newtonian force, not the Minkowskian, has been considered as *the* force, and the magnitude $mc^2(\gamma - 1)$, whose rate of change has been equal to $(\mathbf{Np})$, as the (kinetic) energy of the particle.

This procedure of transferring the transformational properties from certain physical magnitudes to others of the same kind is an important feature of the theory of relativity.

* This fixes, of course, only the transformational properties of forces of any kind, without obliging us, however, to attribute to all such forces a common electromagnetic origin.

After this short digression of a general nature, let us return to our electromagnetic topic.

The formula (II.*a*), obtained above for the force-quaternion F, has nothing to do with the differential equations (I.) of the electromagnetic field. It is only another form of the original formula (II.) for the ponderomotive force. Now, use those differential equations in their quaternionic condensation (I.*a*), that is to say, substitute $C=D\mathbf{L}$. Then the double of the force-quaternion will be

$$2F=D\mathbf{L}\,.\,\mathbf{L}-\mathbf{R}\,.\,D\mathbf{L}, \tag{28}$$

where the dot stands for a separator, stopping the differentiating action of D. This formula, when subjected to a slight, though somewhat peculiar change, will prove to be very convenient for further application. The peculiarity of the formal change alluded to, consists in this, that it requires us to give up an old habit. Hitherto, in conformity with the general convention, we have always used the differential operator D as a 'prefactor,' *i.e.* acting forward only, just as an ordinary scalar differentiator, such as $\partial/\partial t$, is used. Now, the position of a scalar being a matter of indifference, it would be utterly useless and extravagant to write $\partial/\partial t$, for instance, behind the scalar or vector function to be differentiated; for such expressions would mean just the same as $\dfrac{\partial s}{\partial t}$ or $\dfrac{\partial \mathbf{v}}{\partial t}$. But the case is different when the differentiator has the nature of a vector, as the Hamiltonian ∇, or of a quaternion, as D. Since the multiplication of vectors, and more generally of quaternions, is non-commutative, we obviously deprive ourselves of possible advantages if we limit the rôle of quaternionic differential (or other) operators to that of prefactors. Henceforth, therefore, we shall use D as an operator acting *both forward and backward*,* *i.e.* as both a prefactor and a postfactor, and we shall, for instance, write

$$\mathbf{R}[D]\mathbf{L}=\mathbf{R}D\,.\,\mathbf{L}+\mathbf{R}\,.\,D\mathbf{L}, \tag{29}$$

where the dots stop D's differentiating power, and where the brackets (which could also be omitted) are used for better emphasizing the

* To cut short any justification of this departure from convention we could repeat here Oliver Heaviside's words, who, in a similar situation, says simply: 'A cart may be pulled or pushed.' Then, as regards non-differential operators, we have learned long ago from J. W. Gibbs to employ linear vector~~s~~ as both *postfactors* and *prefactors*.
operators

bilateral action of the enclosed operator. The only thing to be still explained in this symbolism is the meaning of $\mathbf{R}D$, which is unusual inasmuch as the operator D follows the operand. Now, if D were an ordinary quaternion, that is a quaternionic magnitude, with s, $\mathbf{v}$ as its scalar and vector parts, we should have, by elementary rules,

$$\mathbf{R}D=\mathbf{R}s+\mathrm{V}\mathbf{R}\mathbf{v}-(\mathbf{R}\mathbf{v})=s\mathbf{R}-\mathrm{V}\mathbf{v}\mathbf{R}-(\mathbf{v}\mathbf{R}).$$

Writing therefore $\partial/\partial l$ instead of s and ∇ instead of $\mathbf{v}$, the plain meaning of $\mathbf{R}D$ will be

$$\mathbf{R}D=\frac{\partial\mathbf{R}}{\partial l}-\mathrm{V}\nabla\mathbf{R}-(\nabla\mathbf{R})=\frac{\partial\mathbf{R}}{\partial l}-\mathrm{curl}\,\mathbf{R}-\mathrm{div}\,\mathbf{R}.$$

This settles the question. Notice that $D\mathbf{R}$ could not be used for relativistic purposes, since $\mathbf{R}$ is right-handed.

Now, to see the utility of $\mathbf{R}D$, return to (1.b), by which $D_c\mathbf{R}=C_c$. Take the conjugate of each side, and remember that $\mathbf{R}_c=-\mathbf{R}$. Then, by the rule of conjugate of a product,

$$-\mathbf{R}D=C,$$

and consequently, by (1.a),

$$D\mathbf{L}=-\mathbf{R}D,$$

and, substituting this in (28),

$$2F=-\mathbf{R}D\,.\,\mathbf{L}-\mathbf{R}\,.\,D\mathbf{L}=-\mathbf{R}[D]\mathbf{L}.$$

In this way we obtain the required short expression for the *force-quaternion*, in terms of the electromagnetic bivectors,

$$F=-\tfrac{1}{2}\mathbf{R}[D]\mathbf{L}. \qquad (11.b)$$

Thus, $\mathbf{R}[\ ]\mathbf{L}$, when applied to D, or more correctly, when exposed to the bilateral differentiating action of D, gives the force-quaternion. We shall see in the next chapter that the same operator $\mathbf{R}[\ ]\mathbf{L}$, when applied to an ordinary vector, *e.g.* the normal of a surface-element, will give us the corresponding stress, and, when applied to a scalar, the density and the flux of electromagnetic energy.

As regards the matrix-equivalents of our bivectors and quaternionic equations, it seemed preferable, for the sake of avoiding any possible confusion, not to insert them in the text of this chapter. Some of these equivalents are given in **Note 6**, which, together with our previous remarks on matrices (Chap. V.), will perhaps be found sufficient.

NOTES ON CHAPTER VIII.

Note 1 (to page 206). Take first the case $\rho=0$, that is to say, consider the equations (I.) outside the charges. Measure x along $\mathbf{v}$, the velocity of S' relative to S. Then

$$\frac{\partial}{c\partial t}=\gamma\frac{\partial}{c\partial t'}-\beta\gamma\frac{\partial}{\partial x'},\quad \frac{\partial}{\partial x}=\gamma\frac{\partial}{\partial x'}-\beta\gamma\frac{\partial}{c\partial t'},\quad \frac{\partial}{\partial y}=\frac{\partial}{\partial y'},\quad \frac{\partial}{\partial z}=\frac{\partial}{\partial z'},$$

and the equations

$$\frac{1}{c}\frac{\partial E_1}{\partial t}=\frac{\partial M_3}{\partial y}-\frac{\partial M_2}{\partial z} \tag{a}$$

and

$$\operatorname{div}\mathbf{E}\equiv\frac{\partial E_1}{\partial x}+\frac{\partial E_2}{\partial y}+\frac{\partial E_3}{\partial z}=0$$

will be transformed into

$$\gamma\frac{\partial E_1}{c\partial t'}-\beta\gamma\frac{\partial E_1}{\partial x'}=\frac{\partial M_3}{\partial y'}-\frac{\partial M_2}{\partial z'}$$

and

$$\gamma\frac{\partial E_1}{\partial x'}-\beta\gamma\frac{\partial E_1}{c\partial t'}=-\frac{\partial E_2}{\partial y'}-\frac{\partial E_3}{\partial z'}.$$

Take the sum of the first and β times the second of these equations. Then the result will be

$$\frac{1}{c}\frac{\partial E_1}{\partial t'}=\gamma\frac{\partial}{\partial y'}(M_3-\beta E_2)-\gamma\frac{\partial}{\partial z'}(M_2+\beta E_3).$$

Thus the form of the equation (a) reappears. Treat similarly the remaining of the equations contained in (I.). Then the whole of these equations, with $\rho=0$, will reappear in dashed letters, thus:

$$\frac{1}{c}\frac{\partial E_1'}{\partial t'}=\frac{\partial M_3'}{\partial y'}-\frac{\partial M_2'}{\partial z'},\ \text{etc.},$$

where

$$E_1'=\psi(v)\,.\,E_1,\quad E_2'=\psi(v)\,.\,\gamma(E_2-\beta M_3),\quad E_3'=\psi(v)\,.\,\gamma(E_3+\beta M_2),$$
$$M_1'=\psi(v)\,.\,M_1,\quad M_2'=\psi(v)\,.\,\gamma(M_2+\beta E_3),\quad M_3'=\psi(v)\,.\,\gamma(M_3-\beta E_2),$$

the common factor $\psi(v)$ being thus far an indeterminate function of v, which for $v=0$ reduces to unity. But solving the last six equations with respect to the non-dashed components and claiming mutually equal rights for the two systems, S and S', we obtain at once

$$\psi(v)\,.\,\psi(-v)=1,$$

and, for reasons of symmetry,

$$\psi(-v)=\psi(v),$$

so that

$$\left.\begin{aligned}E_1'&=E_1,\quad E_2'=\gamma(E_2-\beta M_3),\quad E_3'=\gamma(E_3+\beta M_2)\\ M_1'&=M_1,\quad M_2'=\gamma(M_2+\beta E_3),\quad M_3'=\gamma(M_3-\beta E_2),\end{aligned}\right\} \tag{b}$$

and these are the required formulae of transformation, identical with (7*b*) of this chapter.

Next, pass to the general case of $\operatorname{div}\mathbf{E}=\rho\neq 0$. Bring in the omitted terms ρp_1, etc., the components of $\rho\mathbf{p}$, and, by means of the addition theorem of velocities, express $\mathbf{p}$ in terms of $\mathbf{p}'$ and $\mathbf{v}$. Then the whole of the general equations collected under (I.) will reappear in dashed letters, thus:

$$\frac{1}{c}\frac{\partial E_1'}{\partial t'}+\rho' p_1'=\frac{\partial M_3'}{\partial y'}-\frac{\partial M_2'}{\partial z'}, \text{ etc.,}$$

where

$$\rho'=\operatorname{div}'\mathbf{E}'=\gamma\left(1-\frac{vp_1}{c^2}\right)\rho,$$

or

$$\rho'=\gamma[1-\frac{1}{c^2}(\mathbf{vp})]\,\rho, \tag{c}$$

and where the components of $\mathbf{E}'$, $\mathbf{M}'$ are still connected with those of $\mathbf{E}$, $\mathbf{M}$ by the above formulae (*b*). The details of calculation, similar to those for $\rho=0$, may be left as an exercise for the reader. By working it out fully he will convince himself best of the advantages of shortness and simplicity offered by the quaternionic method employed for the same purposes in the text of the chapter.

Note 2 (to page 209). The difference of the magnetic energy U_m and the electric energy U_e,

$$L=U_m-U_e=\frac{1}{2}\int(M^2-E^2)dS, \tag{a}$$

has been called **the Lagrangian function**, because it has been remarked that the fundamental electronic equations, (I.) and (II.), can be condensed into a single variation-formula having the structure of Hamilton's Principle (or the principle 'of least action'), $\delta\int_{t_1}^{t_2}\ldots=0$, in which precisely that difference of the two kinds of energy appears (along with other possible terms) under the sign of integration. This result is hardly more than a purely formal condensation of the original equations. And since some authors have attributed to it an exaggerated mechanical or dynamical significance, it may be well to give here a short sketch of the bare result and of the method by which it is usually obtained.

Consider a region of space, bounded by the surface σ, fixed in that system S in which the equations (I.) and (II.) hold. Let $\rho=0$ at each of the points of the surface σ, whose choice is otherwise arbitrary. Let the space region, whose volume-elements will be denoted by dS, contain any system of electrons, or more generally, of charges which may be either free or imprisoned in particles of matter in the ordinary sense of the word. Let the 'virtual displacement' consist of a space displacement $\delta\mathbf{r}$ of matter and electrons and of a local variation $\delta\mathbf{E}$ of the electric vector. Let $\delta\mathbf{r}$ and $\delta\mathbf{E}$ be such continuous functions of time and space, as leave

the charge of each element of matter unchanged. With this assumption, and since $\rho = \text{div}\,\mathbf{E}$, the distribution of the infinitesimal vector

$$\delta'\mathbf{C} \equiv \delta\mathbf{E} + \rho\,\delta\mathbf{r} \qquad (b)$$

will be solenoidal, *i.e.* such that $\text{div}(\delta'\mathbf{C}) = 0$. Let W be the infinitesimal virtual work of the ponderomotive forces *of electromagnetic origin only*, *i.e.* by (II.),

$$W = \int(\mathbf{P}\,\delta\mathbf{r})\,dS = \int\rho(\delta\mathbf{r}[\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{p}\mathbf{M}])\,dS.$$

Then, by the differential electronic equations (I.), and after a long but easy calculation (the details of which, together with the literature of the subject, will be found in Lorentz's article in the *Encyklop. der mathematischen Wissenschaften*, Vol. V_2, pp. 167 *et seq.*; Leipsic, 1904):

$$W = \delta(U_m - U_e) - \frac{d}{dt}(\delta' U_m) - \int(\mathbf{X}\mathbf{n})\,d\sigma, \qquad (c)$$

where $\mathbf{X}$ is the infinitesimal vector

$$\mathbf{X} = \mathrm{V}\mathbf{A}\,\delta\mathbf{M} - \mathrm{V}\mathbf{A}\frac{\partial\,\delta'\mathbf{M}}{\partial t} + \mathrm{V}\mathbf{E}\,\delta'\mathbf{M},$$

$\mathbf{A}$ being the usual vector potential, so that $\mathbf{M} = \text{curl}\,\mathbf{A}$. The symbol δ' denotes the variation which would correspond to a change of the total electric current

$$\mathbf{C} \equiv \frac{\partial\mathbf{E}}{\partial t} + \rho\mathbf{p} = c\,.\,\text{curl}\,\mathbf{M}$$

by $\delta'\mathbf{C}$, the elements of matter being kept fixed. This amounts to defining $\delta'\mathbf{M}$ by $c\,.\,\text{curl}\,\delta'\mathbf{M} = \delta'\mathbf{C}$, so that

$$\begin{aligned}\delta' U_m &= \int(\mathbf{M}\,\delta'\mathbf{M})\,dS = \int(\delta'\mathbf{M}\,.\,\text{curl}\,\mathbf{A})\,dS \\ &= \int(\mathbf{A}\ \text{curl}\ \delta'\mathbf{M})\,dS + \int(\mathbf{n}\mathrm{V}\mathbf{A}\,\delta'\mathbf{M})\,d\sigma \\ &= \frac{1}{c}\int(\mathbf{A}\,\delta'\mathbf{C})\,dS + \int(\mathbf{n}\mathrm{V}\mathbf{A}\,\delta'\mathbf{M})\,d\sigma.\end{aligned}$$

Such then is the value of the variation appearing in the second term of (c). But this only by the way.

Now, let σ expand indefinitely. Then, in virtue of the usual assumption as to the behaviour of the field 'at infinity,' the surface integral in (c) will vanish, and

$$W = \delta(U_m - U_e) - \frac{d}{dt}(\delta' U_m). \qquad (d)$$

On the other hand, if T be the usual kinetic energy of matter and V the potential energy, corresponding to the forces of non-electromagnetic origin (which are supposed to be conservative), we have, by d'Alembert's

principle applied to the ordinary, non-electromagnetic masses $\overline{m}$ of the system (if there be any such masses),

$$W=\Sigma\overline{m}\left(\frac{d^2\mathbf{r}}{dt^2}\delta\mathbf{r}\right)+\delta V=\frac{d}{dt}\,\Sigma\overline{m}\left(\frac{d\mathbf{r}}{dt}\delta\mathbf{r}\right)+\delta V-\delta T.$$

Any relativistic amendment of d'Alembert's principle is here disregarded, of course. Combining the last two equations, integrating from $t=t_1$ to $t=t_2$, and assuming that $\delta\mathbf{r}$ and $\delta\mathbf{E}$ vanish at these limiting time instants, we obtain, finally,

$$\delta\int_{t_1}^{t_2}(U_m-U_e+T-V)\,dt=0, \tag{e}$$

that is to say, Hamilton's Principle, in which to the ordinary kinetic energy the magnetic energy U_m and to the potential energy the electric energy U_e is added. In particular, if the whole energy is electromagnetic, as in Abraham's theory, we have simply

$$\delta\int_{t_1}^{t_2}L\,dt=\delta\int_{t_1}^{t_2}(U_m-U_e)\,dt=0. \tag{e_0}$$

The more general equation (e) corresponds to the broader view held by Lorentz.

Thus, $L=U_m-U_e$ plays the rôle of the Lagrangian function. Conversely, assuming $\partial E/\partial t+\rho\mathbf{p}=c\,.\,\mathrm{curl}\,\mathbf{M}$, with $\rho=\mathrm{div}\,\mathbf{E}$, and $\mathrm{div}\,\mathbf{M}=0$, the remaining fundamental electronic equations, *i.e.*

$$\partial\mathbf{M}/\partial t=-c\,.\,\mathrm{curl}\,\mathbf{E} \quad \text{and} \quad \mathbf{P}=\rho[\mathbf{E}+\frac{1}{c}\mathrm{V}\mathbf{p}\mathbf{M}],$$

can be deduced from (e). For slowly varying motion of the electrons, formula (d) gives at once the ponderomotive forces of electromagnetic origin, corresponding to any set of configurational parameters, in the well-known Lagrangian form.

Remember that what is invariant with respect to the Lorentz transformation is the Lagrangian function *per unit volume*, *i.e.* $\frac{1}{2}(M^2-E^2)$. But since $\gamma_p dS$ and dt/γ_p, and consequently also $dS\,.\,dt$ are invariant, the element of 'action'

$$L\,dt=(U_m-U_e)\,dt$$

is an *invariant*. And so also is the whole 'action' $\int_{t_1}^{t_2}L\,dt$ invariant with respect to the Lorentz transformation. It may be noticed here that this is only a particular instance of a general theorem of relativistic dynamics, obtained by Planck.

Note 3 (to page 211). Differentiating $E_1'=E_1$, $E_2'=\gamma(E_2-\beta M_3)$ and $E_3'=\gamma(E_3+\beta M_2)$ with respect to x', y' and z' and passing to x, y, z, we obtain the formula (c) of **Note 1**, in which $\gamma=\gamma_v$. Thus,

$$\rho'=\gamma_v[1-\frac{1}{c^2}(\mathbf{v}\mathbf{p})]\rho.$$

Now, by the addition theorem of velocities (see Chap. VI., and especially formula (b), p. 169),

$$\gamma_p = \gamma_v \gamma_{p'}\left[1 + \frac{1}{c^2}(\mathbf{vp}')\right],$$

whence, by inversion,

$$\gamma_{p'} = \gamma_v \gamma_p\left[1 - \frac{1}{c^2}(\mathbf{vp})\right].$$

Thus $\rho'/\gamma_{p'} = \rho/\gamma_p$, and since $\gamma_{p'} dS' = \gamma_p dS$,

$$\rho' dS' = \rho\, dS,$$

which is the required verification of the invariance of electrical charge.

Note 4 (to page 215). Using the formula obtained for $\mathbf{g}$ on p. 215, we have, for the electromagnetic momentum of the whole field,

$$\mathbf{G} = \int \mathbf{g}\, dS = \frac{v}{c^2}\int [E^2\mathbf{u} - E_1\mathbf{E}]dS,$$

where $\mathbf{u}$ is the unit of $\mathbf{v}$ and E_1 the longitudiual component of $\mathbf{E}$. If $\mathbf{E}_t$ is the transversal part of $\mathbf{E}$, the bracketed terms may be written

$$(E^2 - E_1^2)\mathbf{u} - E_1\mathbf{E}_t,$$

and since the field is, in the case under consideration, symmetrical round $\mathbf{u}$, the transversal terms cancel one another in the process of integration, so that

$$\mathbf{G} = \frac{\mathbf{v}}{c^2}\int (E^2 - E_1^2) dS = G\mathbf{u}.$$

For a Lorentz electron of homogeneous surface-charge,

$$\mathbf{E} = \frac{e\gamma}{4\pi r'^3}\mathbf{r}, \quad r' \geqq R,$$

and $E = 0$ inside the electron. Writing, therefore, $r^2 - x^2 = s^2$, we have

$$G = \frac{v}{c^2}\left(\frac{e\gamma}{4\pi}\right)^2 \int \frac{s^2}{r'^6} dS,$$

where the integral is to be taken throughout the S-space lying outside the ellipsoid $r' = (\gamma^2 x^2 + s^2)^{\frac{1}{2}} = R$. But since this ellipsoid is, for the S'-standpoint, a sphere of radius R, it is easier, of course, to perform the integration in the S'-space. Thus, remembering that $s = s'$ and $dS = dS'/\gamma$ (or that the functional determinant of x, y, z with respect to x', y', z' is $1/\gamma$),

$$\int \frac{s^2}{r'^6} dS = \frac{1}{\gamma}\int \frac{s'^2}{r'^6} dS' = \frac{1}{\gamma}\int \frac{\sin^2\theta'}{r'^4} dS'$$

$$= \frac{8\pi}{3\gamma}\int_R^\infty \frac{dr'}{r'^2} = \frac{8\pi}{3R\gamma},$$

so that

$$G=\frac{e^2\gamma}{6\pi c^2 R}v$$

and

$$\mathbf{G}=G\mathbf{u}=\frac{e^2\gamma}{6\pi c^2 R}\mathbf{v},$$

which is the required formula.

Note 5 (to page 218). Let $\mathbf{A}$, $\mathbf{B}$ be a pair of real vectors and $\mathbf{A}-\iota\mathbf{B}$ a left-handed physical bivector, *i.e.* such that

$$\mathbf{A}'-\iota\mathbf{B}'=Q_c[\mathbf{A}-\iota\mathbf{B}]Q=Q_c\mathbf{A}Q-\iota Q_c\mathbf{B}Q.$$

This splits into

$$\left.\begin{aligned}\mathbf{A}'&=\text{re. } Q_c\mathbf{A}Q-\iota\,.\,\text{imag. } Q_c\mathbf{B}Q\\ \text{and}\qquad \iota\mathbf{B}'&=\iota\,.\,\text{re. } Q_c\mathbf{B}Q-\text{imag. } Q_c\mathbf{A}Q,\end{aligned}\right\}\qquad (a)$$

where re. and imag. stand for 'real part of' and 'imaginary part of.' Now, since Q has a real vector and an imaginary scalar, and since Q_c is the conjugate of Q, it is obvious that

$$\text{re. } Q_c\mathbf{A}Q=\text{re. } Q\mathbf{A}Q_c,$$

$$\text{imag. } Q_c\mathbf{A}Q=-\text{imag. } Q\mathbf{A}Q_c,$$

and similarly for $\mathbf{B}$. Therefore, by (a),

$$\begin{aligned}\mathbf{A}'+\iota\mathbf{B}'&=\text{re. } Q\mathbf{A}Q_c+\iota\,.\,\text{re. } Q\mathbf{B}Q_c+\text{imag. } Q\mathbf{A}Q_c+\iota\,.\,\text{imag. } Q\mathbf{B}Q_c\\ &=Q\mathbf{A}Q_c+\iota Q\mathbf{B}Q_c=Q[\mathbf{A}+\iota\mathbf{B}]Q_c,\end{aligned}$$

that is to say, $\mathbf{A}+\iota\mathbf{B}$ is a right-handed bivector. Q.E.D.

Note 6 (to page 223). Our physical bivector is equivalent to Minkowski's *space-time vector of the second kind* and to Sommerfeld's *six-vector*. Minkowski represents this world-vector by an '*alternating*' matrix

$$h=\begin{vmatrix} 0, & h_{12}, & h_{13}, & h_{14}\\ h_{21}, & 0, & h_{23}, & h_{24}\\ h_{31}, & h_{32}, & 0, & h_{34}\\ h_{41}, & h_{42}, & h_{43}, & 0\end{vmatrix}\qquad (h_{\kappa\iota}=-h_{\iota\kappa}),$$

subjected to the condition that

$$h'=\bar{A}hA,$$

A being the same matrix as in (36) or (40), Chap. V., and $\bar{A}$ the transposed of A. The analogy between $\mathbf{L}'=Q_c\mathbf{L}Q$ and the last transformation formula is seen at a glance. But the multiplication by a quaternion is actually less troublesome than the application of a matrix of 4×4 constituents.

The matrix h is built up of six independent constituents (not counting the diagonal which is always the same). Out of these six constituents three, not containing the index 4, are real, and the remaining three imaginary:

$$h_{23},\ h_{31},\ h_{12} \text{ real,}$$
$$h_{14},\ h_{24},\ h_{34} \text{ imaginary.}$$

Along with h, Minkowski uses the corresponding '*dual*' matrix which he denotes by h^*, and which is again an alternating matrix, *e.g.*

$$h^* = \begin{vmatrix} 0, & h_{34}, & h_{42}, & h_{23} \\ h_{43}, & 0, & h_{14}, & h_{31} \\ h_{24}, & h_{41}, & 0, & h_{12} \\ h_{32}, & h_{13}, & h_{21}, & 0 \end{vmatrix}.$$

This is transformed like h. The product of both matrices,

$$h^*h = h_{32}h_{14} + h_{13}h_{24} + h_{21}h_{34}, \tag{a}$$

which is also the square root of $\det h$, and

$$h_{23}^2 + h_{31}^2 + h_{12}^2 + h_{14}^2 + h_{24}^2 + h_{34}^2 \tag{b}$$

are invariant with respect to the Lorentz transformation. Both of these invariants are contained in the square of our physical bivector.

Let, in particular,

$$\left.\begin{aligned} h_{23} &= M_1, & h_{31} &= M_2, & h_{12} &= M_3 \\ h_{14} &= -\iota E_1, & h_{24} &= -\iota E_2, & h_{34} &= -\iota E_3. \end{aligned}\right\} \tag{c}$$

Then the matrix h will correspond to the electromagnetic bivector $\mathbf{L} = \mathbf{M} - \iota\mathbf{E}$. (In Sommerfeld's four-dimensional language we should say that the magnetic components are projections of the six-vector h upon the planes yz, zx, xy, and $-\iota$ times the electric components the projections of h upon the planes xl, yl, zl.) With this particular meaning of h the matrix form of the electronic differential equations (I.) consists of the equations

$$\left.\begin{aligned} \operatorname{lor} h &= -s \\ \operatorname{lor} h^* &= 0, \end{aligned}\right\} \tag{d}$$

the former embodying the first pair and the latter the second pair of the equations (I.). Here s is the current-matrix,

$$s = \rho \left| \frac{p_1}{c}, \ \frac{p_2}{c}, \ \frac{p_3}{c}, \ \iota \right|,$$

corresponding to the current-quaternion $C = \rho(\iota + \mathbf{p}/c)$. Both of the equations (d) are contained in our $D\mathbf{L} = C$. The two invariants (a), (b) become, in virtue of (c),

$$M^2 - E^2 \quad \text{and} \quad \iota(\mathbf{EM}).$$

Both of these are contained in $\mathbf{L}^2$. The ponderomotive force $\mathbf{P}$, (II.), and its activity are given by the matrix $-sh$. In fact, taking the product of s into h, by the rule given in the Note to Chap. V., we obtain

$$\frac{1}{\rho}sh = \left| -\frac{p_2}{c}M_3 + \frac{p_3}{c}M_2 - E_1, \text{ etc.}, \quad -\iota\frac{p_1}{c}E_1 - \iota\frac{p_2}{c}E_2 - \iota\frac{p_3}{c}E_3 \right|,$$

i.e.

$$-sh = \left| P_1,\ P_2,\ P_3,\ \frac{\iota}{c}(\mathbf{Pp}) \right|. \qquad (e)$$

Since $s' = sA$, as in (37), p. 143, and $h' = \bar{A}hA$, we have $s'h' = shA$, showing that the four-dimensional force, per unit volume, is indeed a world-vector of the first kind. Its quaternionic equivalent is $F = \frac{\iota}{c}(\mathbf{Pp}) + \mathbf{P}$, the force-quaternion of this chapter. The expression $\mathbf{R}C - C\mathbf{L}$ in formula (II.a) takes the place of the matrix $2sh$.

CHAPTER IX.

ELECTROMAGNETIC STRESS, ENERGY AND MOMENTUM. EXTENSION TO GENERAL DYNAMICS.

In the preceding chapter we have seen that the fundamental electronic equations are invariant with respect to the Lorentz transformation, and we have obtained for the force-quaternion per unit volume, *i.e.* for

$$F=\frac{\iota}{c}(\mathbf{Pp})+\mathbf{P}, \tag{1}$$

the short formula (11. *b*), p. 223,

$$F=-\tfrac{1}{2}\mathbf{R}[D]\mathbf{L}. \tag{2}$$

Here D is intended to operate on both $\mathbf{R}$ and $\mathbf{L}$, and the only office of the brackets is to remind us of this bilateral differentiation.

We shall now deduce from this formula the electromagnetic stress $\mathbf{f}_n$ together with the density and the flux of energy. All these magnitudes have already been treated in Chap. II. But now, in virtue of (2), they will appear in a form which will disclose at once their transformational properties.

Take first the scalar part of (2). This gives, by (1), and since $S\mathbf{RL}=-(\mathbf{RL})$,

$$\frac{\iota}{c}(\mathbf{Pp})=\frac{1}{2}\frac{\partial}{\partial l}(\mathbf{RL})-\frac{1}{2}\operatorname{div}V\mathbf{RL},$$

or

$$(\mathbf{Pp})=-\frac{\partial u}{\partial t}-\operatorname{div}\mathfrak{P}, \tag{3}$$

where

$$\left.\begin{aligned} u&=\tfrac{1}{2}(\mathbf{RL}) \\ \mathfrak{P}&=\frac{\iota c}{2}V\mathbf{LR}. \end{aligned}\right\} \tag{4}$$

Remembering the meaning of **L** and **R**, the reader will see at once that these are identical with the familiar formulae $u=\frac{1}{2}(E^2+M^2)$, $\mathfrak{P}=cV\mathbf{EM}$. But the above form will better answer our purposes.

Thus, the scalar part of the equation (2) expresses the conservation of energy, giving the *flux of energy* or the Poynting vector $\mathfrak{P}$, along with u, the *density of electromagnetic energy*. Both of these may also be condensed into the full product,

$$\tfrac{1}{2}\mathbf{RL}=-u+\frac{\iota}{c}\mathfrak{P}. \tag{4a}$$

It is hardly necessary to say that this is *not* a physical quaternion.* But the formula recommends itself by its shortness.

Next, consider the vector part of (2). This is, by (1), the ponderomotive force,

$$\mathbf{P}=-\frac{1}{2}\frac{\partial}{\partial l}V\mathbf{RL}-\frac{1}{2}V\mathbf{R}[\nabla]\mathbf{L},$$

or, by the second of (4),

$$\mathbf{P}=-\frac{1}{c^2}\frac{\partial\mathfrak{P}}{\partial t}-\frac{1}{2}V\mathbf{R}[\nabla]\mathbf{L}.$$

Writing $\nabla=\mathbf{i}\partial/\partial x+\mathbf{j}\partial/\partial y+\mathbf{k}\partial/\partial z$, and remembering that both **R** and **L** are to be differentiated, we have

$$V\mathbf{R}[\nabla]\mathbf{L}=\frac{\partial}{\partial x}V\mathbf{RiL}+\frac{\partial}{\partial y}V\mathbf{RjL}+\frac{\partial}{\partial z}V\mathbf{RkL}. \tag{a}$$

On the other hand, if f is a stress-operator, *i.e.* if

$$f\mathbf{n}=\mathbf{f}_n$$

is the pressure, per unit area, on a surface element whose unit normal is **n**, and if we write in particular, as on p. 48,

$$f\mathbf{i}=\mathbf{f}_1,\quad f\mathbf{j}=\mathbf{f}_2,\quad f\mathbf{k}=\mathbf{f}_3,$$

then the corresponding resultant force per unit volume will be

$$\mathbf{i}\left(\frac{\partial f_{11}}{\partial x}+\frac{\partial f_{21}}{\partial y}+\frac{\partial f_{31}}{\partial z}\right)+\mathbf{j}\left(\frac{\partial f_{12}}{\partial x}+\frac{\partial f_{22}}{\partial y}+\frac{\partial f_{32}}{\partial z}\right)+\mathbf{k}\left(\frac{\partial f_{13}}{\partial x}+\frac{\partial f_{23}}{\partial y}+\frac{\partial f_{33}}{\partial z}\right),$$

or

$$-\frac{\partial\mathbf{f}_1}{\partial x}-\frac{\partial\mathbf{f}_2}{\partial y}-\frac{\partial\mathbf{f}_3}{\partial z},$$

* **R** is not the right sort of neighbour for **L**. In fact, $\mathbf{R}'\mathbf{L}'=Q\mathbf{R}Q_c{}^2\mathbf{L}Q$, and similarly, $\mathbf{L}'\mathbf{R}'=Q_c\mathbf{L}Q^2\mathbf{R}Q_c$.

which is exactly of the form of (a). We see, therefore, that

$$\mathbf{P} = -\frac{\partial \mathbf{g}}{\partial t} - \frac{\partial f\mathbf{i}}{\partial x} - \frac{\partial f\mathbf{j}}{\partial y} - \frac{\partial f\mathbf{k}}{\partial z},$$

where $\mathbf{g} = \mathfrak{P}/c^2$, as on p. 51, and where, for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and hence also for any unit vector $\mathbf{n}$,

$$f\mathbf{n} = \mathbf{f}_n = \tfrac{1}{2}\mathrm{V}\mathbf{RnL}.$$

This is the required formula for the stress. Multiplying out the right-hand side, the reader will easily obtain

$$\begin{aligned} \mathbf{f}_n &= \tfrac{1}{2}(\mathbf{RL})\mathbf{n} - \tfrac{1}{2}\mathbf{R}(\mathbf{Ln}) - \tfrac{1}{2}\mathbf{L}(\mathbf{Rn}) \\ &= u\mathbf{n} - \mathbf{E}(\mathbf{En}) - \mathbf{M}(\mathbf{Mn}), \end{aligned}$$

which is the Maxwellian stress, (20), p. 48. But the above form, obtained directly from (2), is more appropriate for our purposes. Again, since the stress is irrotational, or since f is a symmetrical operator, we have $f\mathbf{i} = \mathbf{i}f$, etc., so that we may write, in the last formula for $\mathbf{P}$,

$$\frac{\partial f\mathbf{i}}{\partial x} + \frac{\partial f\mathbf{j}}{\partial y} + \frac{\partial f\mathbf{k}}{\partial z} = \nabla f,$$

where f is to be considered as a dyadic. (See **Note** 1.) Had we used this short form at the beginning, we might have obtained the above formula for f even more directly.

Thus, the vector part of the equation (2) gives for the ponderomotive force the expression

$$\mathbf{P} = -\frac{\partial \mathbf{g}}{\partial t} - \nabla f, \tag{5}$$

where $\mathbf{g}$, *the electromagnetic momentum* per unit volume, and $f\mathbf{n}$, *the stress* for any orientation of $\mathbf{n}$, are determined by

$$\mathbf{g} = \frac{\mathfrak{P}}{c^2} = \frac{\iota}{2c}\mathrm{V}\mathbf{LR} \tag{6}$$

and

$$f\mathbf{n} = \tfrac{1}{2}\mathrm{V}\mathbf{RnL}. \tag{7}$$

On the other hand, the scalar part of (2) contains the principle of energy, (3), and gives for the density and the flux of energy the expressions (4).

In (7) we have the vector part of a product of three vectors.

Now, the scalar part of this ternary product is

$$\mathbf{SRnL} = \mathbf{SRVnL} = -(\mathbf{RVnL}) = (\mathbf{nVRL}),$$

so that, by (4),

$$\tfrac{1}{2}\mathbf{SRnL} = \frac{\iota}{c}(\mathfrak{P}\mathbf{n}).$$

Consequently, the full product will be

$$\tfrac{1}{2}\mathbf{RnL} = \frac{\iota}{c}(\mathfrak{P}\mathbf{n}) + f\mathbf{n}. \tag{7a}$$

It will be convenient to combine this with (4*a*) into one formula. Let σ be a real, but otherwise arbitrary scalar, and let us introduce for the moment the auxiliary quaternion

$$k = \iota\sigma + \mathbf{n}.$$

Adding $\iota\sigma$ times (4*a*) to (7*a*), we have

$$\tfrac{1}{2}\mathbf{R}k\mathbf{L} = \frac{\iota}{c}[(\mathfrak{P}\mathbf{n}) - c\sigma u] + f\mathbf{n} - \frac{\sigma}{c}\mathfrak{P}. \tag{8}$$

This is valid for any k, that is, for any direction of $\mathbf{n}$ and for any value of σ.

Since (2) transforms into itself, *i.e.* into $F' = -\tfrac{1}{2}\mathbf{R}'[D']\mathbf{L}'$ for any legitimate system S', the same thing is true of the equation of energy (3) and of the formula for the ponderomotive force (5). Both are invariant with respect to the Lorentz transformation. Thus we have, in S',

$$(\mathbf{P}'\mathbf{p}') = -\frac{\partial u'}{\partial t'} - \mathrm{div}'\,\mathfrak{P}'$$

and

$$\mathbf{P}' = -\frac{\partial \mathbf{g}'}{\partial t'} - \nabla' f',$$

where $\mathbf{g}' = \mathfrak{P}'/c^2$ and where $\mathfrak{P}'$, u', f' are determined by the previous formulae, *i.e.* also by (8) with dashed letters. Remember that f' is the stress-operator in S', so that if $\mathbf{n}'$ is a unit vector, $f'\mathbf{n}' = f'_{n'}$ is the pressure on a unit area whose normal is $\mathbf{n}'$.

What are the connexions between $\mathfrak{P}'$, u', f' on the one side and $\mathfrak{P}$, u, f on the other side? To answer this question, return to (8). Take for k a physical quaternion, so that

$$k' = \iota\sigma' + \mathbf{N}' = \iota\sigma' + N'\mathbf{n}' = QkQ,$$

i.e.

$$\sigma' = \gamma\left[\sigma - \frac{1}{c}(\mathbf{vn})\right], \quad \mathbf{N}' = \epsilon\mathbf{n} - \frac{1}{c}\gamma\sigma\mathbf{v}, \tag{9}$$

$\mathbf{n}'$ being the unit of $\mathbf{N}'$. Then $\mathbf{R}k\mathbf{L}$ will also be a physical quaternion, $\simeq q$. Denoting, therefore, by (8) the right side of the equation thus numbered, and by (8′) the same expression with dashes, we have

$$\tfrac{1}{2}\mathbf{R}'k'\mathbf{L}' = (8') = Q(8)Q.$$

Writing down $Q(8)Q$ and equating its scalar and vector parts to the scalar and vector parts of (8′), we obtain the two relations

$$\gamma\left[\frac{1}{c}(\mathfrak{P}\mathbf{n}) - \sigma u\right] - \frac{\gamma}{c}[(\mathbf{v}f\mathbf{n}) - \frac{\sigma}{c}(\mathfrak{P}\mathbf{v})] = \frac{1}{c}(\mathfrak{P}'\mathbf{N}') - \sigma' u',$$

$$\epsilon[f\mathbf{n} - \frac{\sigma}{c}\mathfrak{P}] - \frac{\gamma}{c}\left[\frac{1}{c}(\mathfrak{P}\mathbf{n}) - \sigma u\right]\mathbf{v} = f\mathbf{N}' - \frac{\sigma'}{c}\mathfrak{P}',$$

in which $\mathbf{v}$ is the velocity of S' relative to S and ϵ our previous longitudinal stretcher of ratio $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. Now, since these relations hold for any value of σ, take first $\sigma = 0$, and then $\sigma = 1$, and remember that, by (9),

$$\sigma_0' = -\frac{\gamma}{c}(\mathbf{v}\mathbf{n}), \quad \mathbf{N}_0' = \epsilon\mathbf{n},$$

$$\sigma_1' - \sigma_0' = \gamma, \quad \mathbf{N}_1' - \mathbf{N}_0' = -\frac{\gamma}{c}\mathbf{v}.$$

Then of the four relations, obtained in this way, one, containing the $\mathbf{n}$-component of $\mathfrak{P} - f\mathbf{v}$, will turn out to be a consequence of the three others.

These three relations, after a simple rearrangement of terms, and without Cartesian splitting, give us the required relativistic *transformation of the density and the flux of electromagnetic energy and of the stress* in the short form

$$\left.\begin{aligned} &\frac{1}{\gamma^2}u = u' + \frac{2}{c^2}(\mathfrak{P}'\mathbf{v}) + \frac{1}{c^2}(\mathbf{v}f\mathbf{v}) \\ &\frac{1}{\gamma^2}\mathfrak{P} = \frac{1}{\gamma}\epsilon\mathfrak{P}' + \left[\frac{1}{c^2}(\mathbf{v}\mathfrak{P}') + u' + \frac{1}{\gamma}\epsilon f'\right]\mathbf{v} \\ &\frac{1}{\epsilon}f = f\epsilon + \frac{\gamma}{c^2}[\mathfrak{P}' + u'\mathbf{v}](\mathbf{v} \quad + \frac{\mathbf{v}}{c^2}(\epsilon\mathfrak{P}' \quad . \end{aligned}\right\} \qquad (10)$$

The first of these is a scalar equation, the second a vectorial one, while in the last equation the stress-operator f is written as a dyadic; hence the open parentheses. Introducing on both sides any unit

vector $\mathbf{n}$ as operand, and closing the parentheses, we obtain the corresponding pressure $\mathbf{f}_n = f\mathbf{n}$, thus:

$$\frac{1}{\epsilon} f\mathbf{n} = f'\epsilon\mathbf{n} + \frac{\gamma}{c^2}[\mathfrak{P}' + u'\mathbf{v}](\mathbf{vn}) + \frac{\mathbf{v}}{c^2}(\mathfrak{P}' \cdot \epsilon\mathbf{n}).$$

Remember that ϵ is a symmetrical operator, so that $(\epsilon\mathfrak{P}' \cdot \mathbf{n}) = (\mathfrak{P}' \cdot \epsilon\mathbf{n})$.

To obtain the stress in its more familiar form, take the usual system of normal unit vectors, $\mathbf{i}$ along and $\mathbf{j}$, $\mathbf{k}$ at right angles to the direction of motion. Write in turn $\mathbf{n} = \mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and remember that $\epsilon\mathbf{i} = \gamma\mathbf{i}$, $\epsilon\mathbf{j} = \mathbf{j}$, $\epsilon\mathbf{k} = \mathbf{k}$. Then

$$\frac{1}{\gamma\epsilon}\mathbf{f}_1 = \mathbf{f}_1' + \frac{v}{c^2}[\mathfrak{P}' + u'\mathbf{v}] + \frac{\mathbf{v}}{c^2}\mathfrak{p}_1',$$

$$\frac{1}{\epsilon}\mathbf{f}_2 = \mathbf{f}_2' + \frac{\mathbf{v}}{c^2}\mathfrak{p}_2',$$

$$\frac{1}{\epsilon}\mathbf{f}_3 = \mathbf{f}_3' + \frac{\mathbf{v}}{c^2}\mathfrak{p}_3'.$$

Splitting each of the stress vectors $\mathbf{f}_1$, etc., into its three rectangular components along the same set of axes, we obtain nine stress formulae which contract to six, since $f_{12}' = f_{21}'$, etc., and $f_{12} = f_{21}$, etc. Treating similarly the first two of the equations (10), we have for the transformation of stress and of flux and density of energy the ten Cartesian formulae, which were first given by Laue,

$$\left.\begin{aligned}
&f_{11} = \gamma^2\left(f_{11}' + \frac{2v}{c^2}\mathfrak{p}_1' + \beta^2 u'\right); \quad f_{22} = f_{22}'; \quad f_{33} = f_{33}' \\
&f_{23} = f_{23}'; \quad f_{31} = \gamma\left(f_{31}' + \frac{v}{c^2}\mathfrak{p}_3'\right); \quad f_{12} = \gamma\left(f_{12}' + \frac{v}{c^2}\mathfrak{p}_2'\right) \\
&\mathfrak{p}_1 = \gamma^2[(1 + \beta^2)\mathfrak{p}_1' + (u' + f_{11}')v] \\
&\mathfrak{p}_2 = \gamma(\mathfrak{p}_2' + vf_{21}'); \quad \mathfrak{p}_3 = \gamma(\mathfrak{p}_3' + vf_{31}') \\
&u = \gamma^2\left(u' + \frac{2v}{c^2}\mathfrak{p}_1' + \beta^2 f_{11}'\right).
\end{aligned}\right\} \qquad (10a)$$

The transformation formula of $\mathbf{g}$, the electromagnetic momentum per unit volume, which is simply the energy flux divided by c^2, will be

$$\mathbf{g} = \gamma\epsilon\left[\mathbf{g}' + \frac{v}{c^2}\mathbf{f}_1'\right] + \frac{\gamma^2}{c^2}[(\mathbf{g}'\mathbf{v}) + u']\,\mathbf{v}.$$

Applications of the above formulae will be given a little later, when the domain of their validity has been extended to non-electromagnetic actions. Meanwhile, notice that the stress, energy

and momentum, as estimated from the S-standpoint, are each built up of the stress, energy and momentum or energy flux corresponding to the S'-point of view. This entanglement of the various magnitudes, which in classical physics led an independent existence, is characteristic of the theory of relativity. It is a consequence of the way in which time and space are involved in the fundamental Lorentz transformation.

In deducing the formulae (10) of transformation of stress and associated magnitudes, we have used their expressions in terms of the electromagnetic bivectors, as condensed in (8). Our purpose in doing so was to show the properties of the simple operator $\mathbf{R}[\ ]\mathbf{L}$. But, as a matter of fact, these formulae hold quite independently of the particular, electromagnetic meaning of f, u and $\mathbf{g}$ or $\mathfrak{P}/c^2$. They are valid in virtue of (3) and (5) alone (with $\mathfrak{P}=c^2\mathbf{g}$), that is to say, for stresses etc. *of any origin*, electromagnetic or not, *provided that the corresponding ponderomotive force*, *per unit volume*, *and its activity can be represented in the form*

$$\mathbf{P} = -\nabla f - \frac{\partial \mathbf{g}}{\partial t} \tag{A}$$

$$(\mathbf{Pp}) = -\frac{\partial u}{\partial t} - c^2 . \operatorname{div} \mathbf{g}. \tag{B}$$

The proof of this statement is most simply obtained by the matrix method, which in this case is superior to the quaternionic one. Of course, each method has advantages for certain purposes. In fact, consider the symmetrical matrix

$$\mathfrak{S} = \begin{vmatrix} f_{11}, & f_{12}, & f_{13}, & \iota c g_1 \\ f_{21}, & f_{22}, & f_{23}, & \iota c g_2 \\ f_{31}, & f_{32}, & f_{33}, & \iota c g_3 \\ \iota c g_1, & \iota c g_2, & \iota c g_3, & -u \end{vmatrix}, \tag{11}$$

in which $f_{\iota\kappa}=f_{\kappa\iota}$.* Multiply it by, or operate upon it with, the matrix $\operatorname{lor} = \left| \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial l} \right|$, according to the rule given in the Note to Chapter V. Then the result will be

$$-\operatorname{lor} \mathfrak{S} = \left| P_1, \ P_2, \ P_3, \ \frac{\iota}{c}(\mathbf{Pp}) \right|,$$

* In the case of the electromagnetic field there is a simple connexion between the matrix (11) and the alternating matrix representing the vectors **M**, **E**. See **Note 2** at the end of the chapter.

the constituents of the matrix on the right side being exactly those given by (A) and (B). This matrix is the equivalent of the physical quaternion $F = \mathbf{P} + \frac{\iota}{c}(\mathbf{Pp})$. We can therefore use for it the same letter F. Thus, the last equation can be written

$$F = -\operatorname{lor} \mathfrak{S}. \tag{12}$$

To write this for the force-matrix is exactly the same thing as to postulate (A) and (B) for the force and its activity.

Let me observe here that the matrix (11)* can be written considerably shorter, thus:

$$\mathfrak{S} = \begin{vmatrix} f, & \iota c\mathbf{g} \\ \iota c\mathbf{g}, & -u, \end{vmatrix}. \tag{11a}$$

Here one constituent is a linear operator, or, say, a dyadic, $\mathbf{f} = \mathbf{i})\mathbf{f}_1 + \mathbf{j})\mathbf{f}_2 + \mathbf{k})\mathbf{f}_3$, two other constituents are vectors, and the fourth a scalar. But this heterogeneity of the various constituents of one and the same matrix need not alarm us. It seems even to harmonize fully with the original intention of the creation of Cayley, who wished to see his instrument of multiple algebra treated as broadly as possible. The only requirement is that the array should be rectangular. Using the abbreviated form (11a), we have, of course, to use lor, correspondingly, as the matrix of 1×2 constituents: ∇ to be applied scalarly, and $\partial/\partial l$. In this way we obtain

$$-\operatorname{lor} \mathfrak{S} = -\left| \nabla f + \frac{\partial \mathbf{g}}{\partial t}, \quad \frac{\iota}{c}\left(\operatorname{div} c^2\mathbf{g} + \frac{\partial u}{\partial t}\right) \right| = F$$

at once, instead of writing first so many scalar terms and then gathering them together.

But let us return to our subject. We know already that, whatever the nature of the ponderomotive force, F is a physical quaternion, or the matrix F is transformed as $|\mathbf{r}, l|$. And the same thing is true of lor. Thus, if A be the fundamental transforming matrix, as on p. 143, we have

$$F' = FA, \quad \operatorname{lor}' = \operatorname{lor} A,$$

and therefore, by (12),

$$\operatorname{lor} A\mathfrak{S}' = \operatorname{lor} \mathfrak{S} A,$$

whence, remembering that $A\bar{A} = 1$,

$$\mathfrak{S} = A\mathfrak{S}'\bar{A}. \tag{13}$$

*Which is called *Welttensor* by Laue and others, but has no particular name in Minkowski's paper.

Now, substituting here for A the matrix (40), p. 144, remembering that the transposed matrix $\bar{A}$ is obtained from A by a mere change of the sign of β, and multiplying out the right side, the reader will easily convince himself of the identity of (13) with the transformation formulae (10*a*), in which $\mathfrak{P} = c^2\mathbf{g}$. This proves the above proposition, which may be restated as follows:

> If we make with regard to any ponderomotive forces the assumptions (A) and (B), or, which is the same thing, the assumption
>
> $$F = -\operatorname{lor} \mathfrak{S},$$
>
> then the corresponding pressures, etc., are transformed according to (10*a*) or (10), with $\mathfrak{P} = c^2\mathbf{g}$.

It is, of course, an entirely different question whether those assumptions are to be considered as universally valid or not. Assumption (B) is the expression of *the principle of conservation of energy* together with the concepts of its localization and flux, and (A) leads to the principle of *conservation of momentum*, while there is a strong tendency among the relativists to retain both of these principles of classical physics. Thus, M. Abraham uses (12), involving both principles, in his paper on the electrodynamics of ponderable bodies,* and appeals to this equation even in his theory of gravitation, which does not satisfy the principle of relativity, while Laue makes of it the basis of the general dynamics of continuous bodies. On the other hand, according to Minkowski's electrodynamics of moving ponderable bodies, the ponderomotive force and its activity are expressed by that part of the world-vector $F = -\operatorname{lor} \mathfrak{S}$ †, which is normal to the four-velocity Y, *i.e.* by the matrix

$$F + \frac{1}{c^2} Y\bar{F}Y,$$

or, which is the same thing, by the physical quaternion

$$\tfrac{1}{2}\left[F + \frac{1}{c^2} Y F_c Y\right],$$

* *Rend. Circ. Matem. di Palermo*, Vol. XXVIII., 1909, p. 1; *ibid.*, Vol. XXX., 1910, p. 1.

† This $\mathfrak{S}$ is a non-symmetrical matrix of 4×4 constituents, which reduces to (11) for the particular case of empty space. See Chap. X.

and not by $F = -\text{lor}\,\mathfrak{S}$ itself. Now, it is true that Abraham's and Laue's device recommends itself by its simplicity in the case of the general mechanics of ponderable continua; but, on the other hand, Minkowski's device seems to offer advantages for a relativistic theory of gravitation. In fact, a pair of such theories, both satisfying rigorously the principle of relativity, have been recently proposed by Nordström,* in one of which the four-dimensional force is of the form $\text{lor}\,\mathfrak{S}$, while in the other it is given by the part of such a world-vector perpendicular to Y. Now, the latter of these theories is physically simpler, inasmuch as it leads to a rest-mass independent of the gravitation potential, while the former requires the rest-mass to become an exponential function of this potential.

Certainly, then, the principle of relativity does not compel us to attribute to the forms (A) and (B) of ponderomotive force and its activity an universal validity. But it is at any rate interesting to see the consequences of making the assumptions (A), (B) and of accepting, therefore, the formulae (10) also for pressures, momentum and energy of non-electromagnetic nature in any material medium. Once the reader knows expressly the conditions of their validity, there is no danger in doing so.

We shall therefore proceed to give here some consequences of the formulae (10).

Let the system of reference S' be such that there is *no flux of energy* with respect to it, *i.e.* such that $\mathfrak{P}' = 0$, and therefore also $g' = 0$. This will, under a restriction, be the case when S' is the *rest-system* either of the whole material body, if all its parts have the same velocity relative to S, or, more generally, of its volume-element under consideration. We may retain in both cases the symbol $\mathbf{v}$, which will then generally denote the velocity of an element of the body with respect to S. The restriction hinted at consists obviously in supposing that in u' are contained only such kinds of energy as do not flow through the element in question, *e.g.* energy of elastic deformation, energy stored up in the atoms, heat for the case of uniform temperature, and—as Laue adds—'possibly also some new kinds of energy, yet undiscovered.' But to these the electromagnetic energy cannot generally be added, since it may flow even in the rest-system. If, however, we have in S', say, an electrostatic

* G. Nordström, 'Relativitätsprinzip und Gravitation,' *Phys. Zeitschrift*, XIII., 1912, p. 1126. See also M. Behacker, 'Der freie Fall und die Planetenbewegung in Nordströms Gravitationstheorie,' *ibidem*, XIV., 1913, p. 989.

or a magnetostatic field, we can include in u' the density of the corresponding energy, combining at the same time the electric or the magnetic stress with the mechanical one.

Keeping this in mind, and writing $c^2\mathbf{g}$ for $\mathfrak{P}$, we obtain, for the density of energy and of momentum and for the stress, as estimated from the S-point of view, the formulae (10), considerably simplified,

$$\left.\begin{aligned} u &= \gamma^2\left[u' + \frac{1}{c^2}(\mathbf{v}f'\mathbf{v})\right] \\ \mathbf{g} &= \frac{\gamma^2}{c^2}\left[u'\mathbf{v} + \gamma^{-1}\epsilon f'\mathbf{v}\right] \\ f &= \epsilon f'\epsilon + \frac{\gamma^2 u'}{c^2}\mathbf{v}(\mathbf{v} \ . \end{aligned}\right\} \tag{14}$$

In Cartesians, with axes taken along the velocity $\mathbf{v}$ of a particle and at right angles to it, these formulae are, as (10*a*) without the dashed fluxes of energy,

$$\left.\begin{aligned} &f_{11} = \gamma^2(f_{11}' + \beta^2 u'); \quad f_{22} = f_{22}'; \quad f_{33} = f_{33}' \\ &f_{23} = f_{23}'; \quad f_{31} = \gamma f_{31}'; \quad f_{12} = \gamma f_{12}' \\ &g_1 = \frac{\gamma^2 v}{c^2}(u' + f_{11}'); \quad g_2 = \frac{\gamma v}{c^2} f_{12}'; \quad g_3 = \frac{\gamma v}{c^2} f_{13}' \\ &u = \gamma^2(u' + \beta^2 f_{11}'). \end{aligned}\right\} \tag{14a}$$

We may notice in passing that the sum of the diagonal constituents of the matrix $\mathfrak{S}$, *i.e.*

$$f_{11} + f_{22} + f_{33} - u, \tag{15}$$

is always an invariant. With the above choice of axes, we have also separately, by (14*a*) or by (10*a*),

$$f_{11} - u = f_{11}' - u' \quad \text{and} \quad f_{22} = f_{22}', \ f_{33} = f_{33}'.$$

The invariant (15) vanishes in the case of purely electromagnetic Maxwellian stress. But for mechanical stresses its value will in general differ from zero.

In order to understand the physical applications of the formulae (14), we require still a certain explanation. The stress denoted in these formulae by f, and called the 'absolute' stress, must be carefully distinguished from the elastic stress as usually employed, which, in the writings of Abraham, Laue and other authors, is given

the name of **relative stress.** According to Laue, we have to put, for every dynamically complete system, $F=0$, and to write, therefore, at the head of dynamics of continuous bodies, the equation*

$$\operatorname{lor} \mathfrak{S} = 0. \tag{16}$$

This amounts to putting $\mathbf{P}=0$ in (A) and (B), so that

$$\frac{\partial u}{\partial t} + c^2 \operatorname{div} \mathbf{g} = 0, \quad \frac{\partial \mathbf{g}}{\partial t} = -\nabla f. \tag{16a}$$

At the same time it is assumed that the resultant force acting upon any individual portion of the body in question is given by

$$\mathbf{N} = \frac{d\mathbf{G}}{dt}, \tag{17}$$

where $\mathbf{G} = \int \mathbf{g}\, dS$, the integral being taken throughout the volume of that portion. If, therefore, dS is an individual volume-element of the body, the relative stress which we shall denote by p, the symbol of an operator,† will, according to the familiar definition, be given by

$$\frac{d}{dt}(\mathbf{g}\, dS) = -\mathbf{i}\left(\frac{\partial p_{11}}{\partial x} + \frac{\partial p_{21}}{\partial y} + \frac{\partial p_{31}}{\partial z}\right) dS - \mathbf{j} \ldots$$

or

$$\frac{d}{dt}(\mathbf{g}\, dS) = -\left[\frac{\partial \mathbf{p}_1}{\partial x} + \frac{\partial \mathbf{p}_2}{\partial y} + \frac{\partial \mathbf{p}_3}{\partial z}\right] dS, \tag{a}$$

where $\mathbf{p}_1 = p\mathbf{i}$, etc., and where $\dfrac{d}{dt}$ is the *individual* rate of change. On the other hand, the meaning of the absolute stress f is given by the second of (16*a*) or, in expanded form, by

$$\frac{\partial \mathbf{g}}{\partial t} = -\frac{\partial \mathbf{f}_1}{\partial x} - \frac{\partial \mathbf{f}_2}{\partial y} - \frac{\partial \mathbf{f}_3}{\partial z}, \tag{b}$$

where $\dfrac{\partial}{\partial t}$ is the *local* rate of variation, corresponding to constant

* Laue's symbol equivalent to lor in this connexion is Δiv, a four-dimensional 'scalar divergence,' identical with Sommerfeld's Div.

† So that $p\mathbf{n} = \mathbf{p}_n$ will be the pressure, per unit area, upon a surface element whose normal is $\mathbf{n}$, and p_{n1}, p_{n2}, p_{n3} the rectangular components of this pressure. As will be seen presently, p is, unlike f, a non-symmetrical operator.

values of x, y, z. Now, we have, for any orientation of the system of rectangular axes,

$$\frac{d}{dt}(\mathbf{g}\,dS) = [\mathbf{g}\,\mathrm{div}\,\mathbf{v} + \frac{\partial \mathbf{g}}{\partial t} + (\mathbf{v}\nabla)\mathbf{g}]\,dS$$

$$= \left[\frac{\partial \mathbf{g}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{g}v_1) + \frac{\partial}{\partial y}(\mathbf{g}v_2) + \frac{\partial}{\partial z}(\mathbf{g}v_3)\right] dS,$$

and therefore, comparing (a) with (b),

$$\mathbf{p}_1 = \mathbf{f}_1 - \mathbf{g}v_1, \quad \mathbf{p}_2 = \mathbf{f}_2 - \mathbf{g}v_2, \quad \mathbf{p}_3 = \mathbf{f}_3 - \mathbf{g}v_3, \tag{18a}$$

i.e. for any direction of $\mathbf{n}$,

$$\mathbf{p}_n = \mathbf{f}_n - \mathbf{g}(\mathbf{vn}).$$

Omitting the operand $\mathbf{n}$, we may write this result, in terms of the stress-operators themselves,

$$p = f - \mathbf{g}(\mathbf{v} \quad . \tag{18}$$

This is the required connexion between *the relative stress* p and the absolute stress f. Notice that, f being symmetrical or self-conjugate, p is in general non-symmetrical, since $\mathbf{g}$ may differ in direction from $\mathbf{v}$. Thus, for instance, $p_{12} = f_{12} - g_2 v_1$, while $p_{21} = f_{12} - g_1 v_2$. Only when $\mathbf{g} \parallel \mathbf{v}$ does the relative stress become self-conjugate.

Let us now return to (14). Remember that, for the rest-system, $p' = f'$, write down $\mathbf{g}(\mathbf{v}$ by the second of those formulae, and subtract it from the third one. Then the terms containing u' will cancel one another, and the result will be

$$p = \epsilon p' \epsilon - \frac{\gamma}{c^2} \epsilon p' \mathbf{v}(\mathbf{v} \quad ,$$

or, if $\mathbf{i}$ be the unit of $\mathbf{v}$,

$$p = \epsilon p' \epsilon - \beta^2 \gamma \,.\, \epsilon p' \mathbf{i}(\mathbf{i} \quad . \tag{19}$$

Such then is the transformation formula of the relative stress. The reader will find no difficulty in splitting (19) into nine Cartesian equations for p_{11}, p_{12}, etc., especially as this procedure has been illustrated a moment ago by the passage from (14) to (14a). It is interesting to remark that p depends only upon p' and the motion of the element in question, but not upon u', the density of energy. And, whenever $p' = 0$, we have also $p = 0$. This, besides the original definition (a), is the reason why the relative stress p (and not the absolute one) is considered as *the* stress.

The simplest case occurs when the body, viewed from the rest-system, is subjected to what is called a hydrostatic or *isotropic*

pressure (*i.e.* to a pressure which is purely normal and equal for all directions of $\mathbf{n}$), either uniform or varying from point to point. Then the stress-operator p' degenerates into an ordinary scalar, *the pressure* in the more familiar sense of the word.* In this case p' can be written before the stretching operator, so that (19) gives at once

$$p : p' = \epsilon^2 - \beta^2\gamma\epsilon\mathbf{i}(\mathbf{i} \quad = \epsilon^2 - \beta^2\gamma^2\mathbf{i}(\mathbf{i} \quad .$$

Now, $\epsilon^2 = \gamma^2\mathbf{i}(\mathbf{i} + \mathbf{j}(\mathbf{j} + \mathbf{k}(\mathbf{k}$, and $\gamma^2 - \beta^2\gamma^2 = 1$, so that the right side of the last formula is, in Gibbs' terminology, an idemfactor, $\mathbf{i}(\mathbf{i} + \mathbf{j}(\mathbf{j} + \mathbf{k}(\mathbf{k}$, leaving unchanged any operand $\mathbf{n}$ whatever. The result, therefore, is that

$$p = p',$$

or that *isotropic pressure is a relativistic invariant.* This result was first obtained by Planck† from thermodynamical considerations aided by the principle of relativity, then by Sommerfeld‡ from what he believed to be a purely geometric enunciation of the behaviour of four-dimensional vectors and their projection, and, finally, by Laue, whose method has been here adopted. The reader will find it worth his while to compare the latter with the two former methods, and is for that purpose referred to the papers of Planck and Sommerfeld just quoted.

So much as regards the stress and its transformation. Next, consider u and $\mathbf{g}$, the densities of energy and of momentum for which the first pair of (14) hold. In these formulae we have only to substitute the identity $f' = p'$. Thus, taking $\mathbf{i}$ along the direction of motion of the given element of the body, we have in general, that is to say, for any elastic stress p',

$$u = \gamma^2[u' + \beta^2 p_{11}'] \tag{20}$$

and

$$\mathbf{g} = \frac{\gamma^2}{c^2}[u'\mathbf{v} + \gamma^{-1}\epsilon p'\mathbf{v}], \tag{21}$$

where $p'\mathbf{v}$ is the same thing as $v\mathbf{p}_1'$, of course.

Let dS' be the rest-volume of an element of the body, and consequently $dS = dS'/\gamma$ its S-volume. Then we shall have for the

* Reckoned positive if pressure proper, and negative if tension proper, as before.

† M. Planck, 'Zur Dynamik bewegter Systeme,' *Ann. der Physik*, Vol. XXVI., 1908, pp. 1-34.

‡ *Ann. der Physik*, Vol. XXXII., 1910, p. 775.

energy of that individual element, as estimated from the S-point of view,

$$u\,dS = \gamma(u' + \beta^2 p_{11}')\,dS'.$$

To obtain the whole energy U, this is to be integrated throughout the body. Generally speaking, there will be no simple relation between U and U'. For, even if u' and the stress were constant throughout the body, the value of β and also the direction of $\mathbf{v}$ may change from point to point. And if but one particle of the body moves with varying velocity, then the velocity will also, as a rule, vary from particle to particle. Let us suppose, however, that this heterogeneity of the inner state (u', p') and of the motion of the body can be neglected. Then, if V and V' be the volumes of the whole body from the two standpoints, its total energy, as estimated by the S-observers, will be

$$U = \gamma(U' + \beta^2 p'_{11} V'). \tag{20a}$$

We shall return to this formula presently, in order to compare the difference $U - U'$ with the expression of kinetic energy given, for the simplest particular case, in Chapter VII.

Treating similarly the equation (21), and making the same assumption of homogeneity, or considering the whole body as a particle, we have, for its total momentum,

$$\mathbf{G} = \frac{\gamma}{c^2}[U' + V' . \gamma^{-1}\epsilon p']\mathbf{v}. \tag{21a}$$

We have seen in Chap. VII., formula (24), that, according to Minkowski's dynamics of a particle, the momentum of the particle would be simply γm times its velocity, where m, the *rest-mass* of the particle, is an ordinary scalar magnitude. Thus, according to that manner of treatment, the momentum would always coincide in direction with the velocity. This isotropic behaviour of the rest-mass appears now as the simplest particular case of formula (21a), which holds for a particle conceived as the limit of an extended body. We can still write

$$\mathbf{G} = \gamma m \mathbf{v},$$

but now m, instead of being a simple scalar, will be a linear vector operator, *e.g.*

$$m = \frac{U'}{c^2} + \frac{V'}{c^2\gamma}\epsilon p', \tag{22}$$

so that the momentum will generally differ in direction from the velocity.

The first part of m is an ordinary scalar, namely

$$U'/c^2.$$

This is the expression of the famous *inertia of energy* which, as a consequence of the principle of relativity, has been enunciated by Einstein.* If a body gains or loses n ergs of energy, say, in the form of heat, then we have to look for an increase or diminution of its rest-mass by $\frac{n}{9}10^{-20}$ grams. The second part of m is due to the stress. Since p' is, in general, an operator, this part of m will also be an operator. It will be remembered that p', being identical with the original f', is self-conjugate. The stress, therefore, will have three mutually perpendicular principal axes. Let these be represented by the unit vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, each of which can be taken, of course, in both its positive and negative sense. And let us denote the corresponding principal pressures, which are ordinary scalars, by p_a', p_b', p_c'. Then, if $\mathbf{v}$ is along $\mathbf{a}$, for instance, we shall have

$$\frac{1}{\gamma}\epsilon p'\mathbf{a}=\frac{1}{\gamma}\epsilon\mathbf{a}\,.\,p_a'=\mathbf{a}\,.\,p_a',$$

since $\epsilon\mathbf{a}=\gamma\mathbf{a}$. Similarly, if the body happens to move along $\mathbf{b}$ or $\mathbf{c}$. Thus, *the principal axes of the mass-operator m coincide with the principal axes of the stress.*† The corresponding principal values of the rest-mass are

$$\left.\begin{aligned} m_a&=\frac{1}{c^2}(U'+V'p_a') \\ m_b&=\frac{1}{c^2}(U'+V'p_b') \\ m_c&=\frac{1}{c^2}(U'+V'p_c'). \end{aligned}\right\}\qquad(22a)$$

*Cf. Einstein's papers in *Ann. der Physik*, Vol. XVIII., 1905, p. 639, Vol. XX., 1906, p. 627, but especially 'Ueber die vom Relativitätsprinzip geforderte Trägheit der Energie,' *ibid.*, Vol. XXIII., 1907, p. 371. Independently of the principle of relativity, the inertia of energy, in the case of radiation, appears in a valuable paper of K. v. Mosengeil, *Ann. der Physik*, Vol. XXII., 1907, p. 867. The history of this concept can, of course, be traced a long way farther back. Its origin can be looked for in Maxwell's pressure of light, and in connexion with this many English physicists spoke about 'momentum carried by light waves' a long time before the theory of relativity arose.

†This coincides with Herglotz's result obtained by a different method: *Ann. der Physik*, Vol. XXXVI., 1911, p. 493. The reader will find in this beautiful paper a systematic development of relativistic mechanics of deformable bodies.

The momentum is parallel to the velocity of the body when and only when it happens to move along one of its principal stress-axes.

Notice that, by what has been said, this anisotropy would be a property of the rest-mass itself. When, therefore, we pass to consider the acceleration of such a body, or particle, in relation to the moving force, according to the equation of motion

$$\frac{d}{dt}\gamma m\mathbf{v} = \mathbf{N}, \tag{23}$$

we can no longer express the inertial behaviour of the body in terms of a 'longitudinal' and a 'transversal' mass, as in Chapter VII. The axial symmetry produced round $\mathbf{v}$ in that comparatively simple case was due to the assumption of a scalar rest-mass. The case now before us is much more complicated. Even if the inner state of the body is supposed to remain invariable, a full description of acceleration in connexion with force requires a linear vector operator, involving *six* scalar inertial coefficients. The dynamics of translational motion of such a body is, obviously, entangled with the dynamics of its rotations. Unlike classical mechanics, these two kinds of motion cannot, rigorously speaking, be treated separately. It can be shown, by considering the moment of momentum, that to maintain such a body in uniform rectilinear motion, a certain couple is required. Only when the constant vector-velocity $\mathbf{v}$ of the body coincides in direction with one of its principal stress-axes, would the moment of this couple vanish. Again, suppose that there is no impressed resultant force, *i.e.* that $\mathbf{N} = 0$. Then the momentum will be constant in both size and direction relative to S, say, equal $\mathbf{C}$, and

$$\gamma\mathbf{v} = m^{-1}\mathbf{C}.$$

If, therefore, the body rotates together with its stress-axes, the motion of translation will not be uniform and even not rectilinear. Notwithstanding the absence of a resultant S-force the body may move with varying velocity relative to the framework S. And it will do so if, for instance, its initial velocity does not coincide in direction with one of the principal stress-axes and if the couple mentioned above is not applied. But we cannot dwell any longer upon this curious subject.

All that has just been said with regard to the anisotropy of rest-mass has, at least for the time being, merely a theoretical interest. In fact, nobody has ever observed in translational inertia

any departure from isotropy. On the other hand, it must be confessed that no phenomena of this kind have been sought for expressly and that direct comparisons of inert masses (*i.e.* apart from gravity) could not easily be made more accurate than to one in ten or hundred thousand parts. One thing, at any rate, seems certain: If the above formulae are accepted, we cannot reasonably hope to produce observable anisotropy of mass by artificial pressures or tensions in any lump of matter. For, according to (22*a*), hundreds of atmospheres appropriately applied would produce a departure from isotropy of mass amounting only to $10^2 . 10^6 c^{-2} \doteq 10^{-13}$ of a gram per cubic centimetre. But for all that we know there might be anisotropy of inertia in natural crystals, corresponding to some enormous 'latent stresses.' And to embody such stresses into p' seems no less, and no more, legitimate than to condense in U' so much 'latent energy' as is necessary to account for the observable mass of a body. But, apart from any theory, experiments on crystals seem worth trying, whether to reveal some traces of anisotropic inertia or to push it below a numerically definite limit.*

Of course, if it is assumed that the stresses represented by p' are, under all circumstances, only of the order of manifest tensions and pressures known as such from experience, then the influence of the differences $p_a' - p_b'$, etc., upon inertia will be far too small to be ever detected. But if so, then there will be also no sensible contribution of stress to inertia at all. Such, in fact, is the prevailing opinion.

According to this opinion the stress-term in (22), (21) and, for slow motion, *a fortiori* in (20), where it appears with the coefficient β^2, can be omitted for all ordinary material bodies. But the case is

*In connexion with this subject, Prof. A. W. Porter of University College, London, draws my attention to experiments made by Poynting and Gray, who tested for anisotropy of gravitation between two quartz spheres (*Phil. Trans.*, 192, 1899, A. p. 245; cf. also Poynting and Thomson's *Text-Book of Physics*, *Properties of Matter*, London, 1909, p. 48). Their results showed that this anisotropy could not amount in one case to more than one part in 2800, and in another case to more than one part in 16000. On the other hand, proportionality between mass and gravitation, first tested by Newton in his pendulum experiments and carried to further refinement by Bessel, has been more recently shown by Eötvös (*Math. und naturwiss. Berichte aus Ungarn*, Vol. VIII., 1890) to be true to one part in ten millions, in the case of isotropic bodies at least. So far as we know, experiments of this kind have not yet been made with crystalline bodies, but are now under consideration at University College.

different, of course, when the energy and the stress are purely electromagnetic, when the 'body' becomes simply a region of space containing an electromagnetic field. Under these circumstances the part played by p' is no longer negligible, unless we wish to neglect the whole mass m, and therefore also the whole momentum. In fact, not only then are the pressures or tensions p_a', etc., of the same order as the density u' of electromagnetic energy, but some of them can even wholly annul the contribution of energy to mass. Let, for instance, the field in S' be a homogeneous electrostatic field $\mathbf{E}' = \mathbf{const.}$, such as is contained between the plates (discs) of a plane vacuum-condenser, far enough from the edges of the plates. Then $u' = \frac{1}{2}E'^2$, and if $\mathbf{a}$ be taken along the axis of the condenser or along the Faraday tubes, p_a', being a tension proper, is equal to $-\frac{1}{2}E'^2$, while p_b', p_c', being pressures proper, are each equal to $\frac{1}{2}E'^2$. Therefore, by ($22a$),

$$m_b = m_c = \frac{2U'}{c^2} = \frac{E'^2 V'}{c^2},$$

while

$$m_a = 0.$$

Thus the condenser, apart from the plates, has equal rest-masses in all transversal directions, while its longitudinal principal rest-mass vanishes altogether. If it is moved along the tubes it has no momentum. This property, which holds separately for each length-element of a Faraday tube, harmonizes with Sir J. J. Thomson's well-known representation. The tubes may be straight, as in the above case, or curved and of varying section. The only condition being that there shall be no flux of energy in S', we can certainly apply the above reasoning to any electrostatic field. Summing up the contributions due to the elements of infinitesimal filaments (with appropriate consideration of their directions), the mass-operator of the whole field can be found. If the field is radial and symmetrical round a point O', as in the case of the Lorentz electron, the mass-operator m degenerates into an ordinary scalar, the rest-mass of the electron, or rather of its whole field. The reader is recommended to prove this in detail, and to compare the result to be thus obtained with the formula of the electromagnetic rest-mass given previously.*

* The above dynamical considerations have also an important bearing upon the theory of the celebrated condenser-experiment of Trouton and Noble (*Proceedings Roy. Soc.*, Vol. LXXII., 1903), in which a second-order moment of rotation on a

Let us now once more return to stresses and energies of any origin. In the simplest case of hydrostatic or *isotropic pressure*, whatever its order of magnitude, our above p' degenerates into an ordinary scalar, so that, in (21*a*), $\gamma^{-1}\epsilon p'\mathbf{v}=\gamma^{-1}\epsilon\mathbf{v}\,.\,p'=\mathbf{v}\,.\,p'$, while, in (20*a*), $p_{11}'=p'$, and therefore

$$\left.\begin{aligned} U&=\gamma(U'+\beta^2 p'V') \\ \mathbf{G}&=\frac{\gamma}{c^2}(U'+p'V')\mathbf{v}. \end{aligned}\right\} \qquad (24)$$

These are Planck's formulae (*loc. cit.*). Since isotropic pressure is an invariant and $V=V'/\gamma$, we have also

$$\chi=U+pV=\gamma(U'+p'V')=\gamma\chi', \qquad (25)$$

where χ', the rest-value of χ, is Gibbs' 'heat function for constant pressure' or **enthalpy.*** The momentum is now in the direction of motion. The mass-operator (22) degenerates into

$$m=\frac{U'+p'V'}{c^2}=\frac{\chi'}{c^2}, \qquad (26)$$

the scalar rest-mass.

Thus, in the case of isotropic stress, the inertial behaviour of the body, or particle, is characterized by a simple scalar, as in Chap. VII. But still the rest-mass will in general vary in time, inasmuch as the inner state of the particle (U', p', V') may undergo changes during its motion. If this is the case, *e.g.* if the enthalpy of the particle varies, then SXY_c does not vanish, or, in other words, the Minkowskian four-force X is no longer perpendicular to the particle's world-line. In fact, instead of equation (20), p. 194, we now have

$$m\frac{dY}{d\tau}+Y\frac{dm}{d\tau}=X,$$

suspended condenser due to the earth's orbital motion was sought for. But a somewhat thorough exposition of this subject would be beyond the limits and purposes of the present volume, and the interested reader must therefore be referred to § 18 of Laue's book already quoted. Here it will be enough to say that the relativistic theory accounts fully for the negative result of the Trouton-Noble experiment.

* The latter name is used by the Dutch school of physical chemists, while the name nearly always used in England is *total heat*.

and consequently, since t' can be written for the proper time τ,

$$\mathrm{S}XY_c = YY_c\frac{dm}{dt'} = -c^2\frac{dm}{dt'},$$

or, by (26),

$$\mathrm{S}XY_c = -\frac{d\chi'}{dt'}.$$

This proves the statement. Developing the left-hand side, by (18), (17*a*), p. 193, we have, in terms of the Newtonian force **N** and the velocity **v** of the particle,

$$(\mathbf{Nv}) = \frac{d}{dt}(mc^2\gamma) - \frac{1}{\gamma^2}\frac{d\chi'}{dt'},^* \tag{27}$$

or also, by (25) and (26),

$$(\mathbf{Nv}) + \frac{1}{\gamma^2}\frac{d\chi'}{dt'} = \frac{d\chi}{dt}. \tag{27a}$$

This is now, instead of (22), p. 194, the equation of energy.

To see its meaning, consider the particular case of constant pressure, or what may be called *isopiestic motion*. Then, if h' be the heat communicated to the particle per unit t'-time,

$$\frac{d\chi'}{dt'} = \frac{dU'}{dt'} + p'\frac{dV'}{dt'} = h', \tag{28'}$$

the heat supply being estimated from the point of view of the system S' in which the particle is instantaneously at rest. Consequently,

$$(\mathbf{Nv}) + \frac{1}{\gamma^2}h' = \frac{dU}{dt} + p\frac{dV}{dt}. \tag{28}$$

The first term on the right is the rate of increase of the total energy of the particle, the second term gives the work done per unit time by the particle in expanding, while $(\mathbf{Nv})$ is the activity of the impressed force, everything being estimated from the S-point of view. If, therefore, (28) is to express the conservation of energy in S, just as (28′) does with respect to S', we have to write for h, the rate of heat supply as estimated from the S-point of view,†

$$h = \frac{h'}{\gamma^2}. \tag{29}$$

* This result can be verified at once by multiplying eq. (23) of the present chapter scalarly by **v**.

† It is scarcely necessary to warn the reader that h is not equal to $d\chi/dt$. It becomes so (for constant pressure) only in the rest-system. Putting in (28) $\mathbf{v}=0$, $\gamma=1$, we obtain (28′).

And, since $dt = \gamma\, dt'$, we have to require that the relativistic connexion between corresponding infinitesimal amounts of heat supplied or withdrawn shall be

$$\delta H = \frac{1}{\gamma}\,\delta H'. \tag{30}$$

This transformation formula agrees entirely with what follows from Planck's thermodynamical investigation. In fact,* one of Professor Planck's most fundamental results is that *entropy is invariant* with respect to the Lorentz transformation,

$$\eta = \eta',$$

and another of his results states that *temperature is transformed like volume*,

$$\theta = \frac{1}{\gamma}\,\theta'.$$

Now, the temperature being here defined in the well-known thermodynamical way, we have, for reversible heat supply, $\delta H' = \theta' d\eta'$, and on the other hand (granting that a process reversible in S' is also reversible from the S-standpoint), $\delta H = \theta\, d\eta$, whence $\delta H = \delta H'/\gamma$.

But, instead of recurring to temperature and the second law of thermodynamics, the transformation formulae (29) and (30) can equally well be considered as consequences of the principle of conservation of energy combined with (28), which in its turn is a consequence of the equation of motion (23) and of the relativistic behaviour of momentum. Whatever the logical order of exposition, the important thing to notice is that the several properties are consistent with one another.

Before leaving the discussion of variable rest-mass, only one more remark. It has been shown in Chap. VIII. that the electromagnetic ponderomotive force per unit volume *plus* ι/c times its activity is a physical quaternion. In agreement with this the total force $\mathbf{N}$ of Chap. VII. had the property that $\gamma[\iota(\mathbf{N}\mathbf{v})/c + \mathbf{N}]$ was a physical quaternion. Both of these were particular instances of a more general property which is now before us. When the moving body receives or gives up heat, or more generally, when its enthalpy

* Cf. Planck's paper quoted on p. 245. Unfortunately, there is in this book no place for an adequate discussion of the foundations of relativistic thermodynamics.

is varying, then the above expression is no longer a physical quaternion. What now continues to be such a quaternion is

$$X=\frac{d}{d\tau}m\,Y=\gamma\left[\frac{dm\gamma\mathbf{v}}{dt}+\iota c\frac{d\gamma m}{dt}\right],$$

or

$$X=\gamma\left[\frac{\iota}{c}\frac{d}{dt}(mc^2\gamma)+\mathbf{N}\right].$$

In Chap. VII. we had simply $\frac{d}{dt}(mc^2\gamma)=(\mathbf{N}\mathbf{v})$, while now we have, instead of this, the equation (27). Thus, in general, for any motion of the body,

$$X=\gamma\left\{\frac{\iota}{c}\left[(\mathbf{N}\mathbf{v})+\frac{1}{\gamma^2}\frac{d\chi'}{dt'}\right]+\mathbf{N}\right\}$$

is a physical quaternion, and more especially, for *isopiestic* motion,

$$X=\gamma\left\{\frac{\iota}{c}[(\mathbf{N}\mathbf{v})+h]+\mathbf{N}\right\}\simeq q. \tag{31}$$

In all such cases, therefore, we have to add to the activity of the impressed force the amount of heat supplied to the body per unit time. This property will reappear, in the next chapter, in connexion with Joule's heat in electrical conductors.

If the enthalpy χ', and therefore also the rest-mass of the body, is kept *constant*, we fall back to the simple case treated in Chapter VII. The activity then becomes, by (27),

$$(\mathbf{N}\mathbf{v})=\frac{d}{dt}(mc^2\gamma), \tag{32}$$

identical with (22), p. 194. Using the form (27*a*), we may also write, equivalently,

$$(\mathbf{N}\mathbf{v})=\frac{d\chi}{dt}=\frac{dU}{dt}+p\frac{dV}{dt}, \tag{32a}$$

which reads: Work done upon the body = increase of its energy *plus* work done by the body in expanding. The corresponding condition

$$\chi'=U'+p'V'=\text{const.}$$

can still be fulfilled in a variety of ways. Thus the motion may be adiabatic as well as isopiestic. Or, we may give up both of these conditions and suppose instead that $V'dp'/dt'$ is just balanced by the (positive or negative) heat supply h'. Or finally, the inner state

of the moving body may be invariable, *i.e.* U', p' as well as V' may be kept constant. But even then the work done by expansion does not disappear from (32*a*) unless the motion is uniform. For, with constant V', we have

$$\frac{dV}{dt} = V' \frac{d\gamma^{-1}}{dt},$$

which expresses the varying FitzGerald-Lorentz contraction. But whatever the way in which χ' is kept constant, we have the same equation of motion as in Chap. VII.,

$$m \frac{d\gamma\mathbf{v}}{dt} = \mathbf{N},$$

and consequently the longitudinal and the transversal masses return to their rights, being again given by

$$m_l = m\gamma^3, \quad m_t = m\gamma,$$

where m has now the explicit meaning

$$m = \frac{\chi'}{c^2} = \frac{U' + p'V'}{c^2}. \tag{33}$$

Finally, if the pressure p', and therefore also p, is assumed to vanish, the equation of energy becomes

$$(\mathbf{N}\mathbf{v}) = \frac{dU}{dt},$$

and the constancy of the rest-mass

$$m = \frac{U'}{c^2}$$

means constancy of the particle's store of energy. In this case the difference between the energies U and U' can be looked at as entirely due to the motion of the particle and called its kinetic energy relative to S. The value of the kinetic energy thus defined is identical with that given on p. 195. In fact, the first of (24) becomes now $U = \gamma U'$, so that

$$\begin{aligned} U - U' &= (\gamma - 1)\, U' = mc^2(\gamma - 1) \\ &= \tfrac{1}{2}mv^2(1 + \tfrac{3}{4}\beta^2 + \tfrac{5}{8}\beta^4 + \ldots). \end{aligned}$$

But, speaking rigorously, 'kinetic energy' is now deprived of the distinct part it played in classical mechanics. And its entanglement with other kinds of energy becomes even more intricate when we pass from this simplest case to any of the preceding ones.

It may be useful to illustrate here the mass formula (33) by a few numerical examples. Thus, taking 2·1 gram calories for what is called the solar constant (energy received from the sun per minute per cm^2. at the earth's mean distance), we have for the sun's total radiation per minute

$$4\pi(1\cdot5\,.\,10^{13})^2\,.\,2\cdot1\,.\,4\cdot2\,.\,10^7 \text{ ergs},$$

so that the diminution of the sun's mass due to radiation would be, per minute, $2\cdot8\,.\,10^{14}$ grams, and per year

$$\delta m = 1\cdot5\,.\,10^{20} \text{ grams.}$$

This seems at first a prodigious loss; but the sun's mass being $2\,.\,10^{33}$ gr., the proportionate loss per year,

$$\frac{\delta m}{m} = \tfrac{3}{4}\,.\,10^{-13},$$

is quite insignificant. Next, take the example adduced by Planck. A mixture of 2 gr. of hydrogen and 16 gr. of oxygen develops in the act of producing water, at ordinary pressure and temperature, $2\cdot9\,.\,10^{12}$ ergs of heat; the corresponding diminution of mass amounts to $3\cdot2\,.\,10^{-9}$ gr., and the proportionate loss due to this intense reaction,

$$\frac{\delta m}{m} = 2\,.\,10^{-10},$$

would again be far too small to be observed. Numbers of similar order would result for other instances of chemical reaction. In short, the 'latent energy' which (if we neglect the contribution due to stress) is to account for mass does not manifest itself in any one of those processes in which atoms are implied as wholes. We are thus driven back to the interior of the old chemist's atom, and have to look for that energy in the disintegration of atoms known in connexion with radioactive phenomena. In fact, if we are to judge from their observed heat-effect only, the amounts of energy developed in such processes exceed immensely all those liberated in ordinary chemical reactions, and Professor Planck seems to see in radioactivity a kind of verification of the energetic theory of inertia. Now, it is true that these processes have disclosed to the physicist atomic stores of energy of a copiousness not even suspected a short time ago. But, notwithstanding their unparalleled vigour, radioactive phenomena reveal but a very minute fraction of the assumed latent energy. Thus, to

quote Planck's own example, one gram-atom of radium would lose through its heat production (30240 gram calories per hour) ·012 miligr. of its mass per year; the proportionate loss, therefore, amounting to

$$\delta m/m = \cdot 5 \,.\, 10^{-7} \text{ per annum},$$

is again too small to be observed.

We may add that the latter δm is small even when compared with the mass disintegrated during the same interval of time. For this amounts, per 225 gr. of radium present, and per year, to $m_{\text{dis}} = 9 \,.\, 10^{-2}$ gr., so that, in round figures,

$$\delta m/m_{\text{dis}} = 10^{-4}.$$

The mass of the disintegrated parent substance reappears sensibly undiminished in the masses of the descendants.

Thus, even radioactive phenomena reveal to us practically nothing of the assumed latent energy $c^2 m$. Its bulk remains as latent as anything ever was. It must, therefore, be confessed that the energetic theory of rest-mass, attractive and promising as it may seem, has for the time being the character of a purely formal reduction of one concept to another. Nobody doubts, of course, that the chemical atoms are themselves exceedingly complicated systems, and that there are therefore many ways left of throwing the chief stores of latent energy upon a host of ultra-atomic entities, electrons or what not. If so, then some spontaneous disintegration, affecting the atomic structure even more profoundly than that which in our days is associated with the name of radioactivity, may induce the gates of those copious stores to open to the human eye. But as yet we have not the least knowledge of such phenomena. It is for this reason we have said that it is equally legitimate to assume latent stresses along with the manifest ones in the mass formulae as to assume latent energies. Both are originally defined only by their variations, in time and in space respectively. And, for the present, both would have a purely formal character.

The above mechanical, and partly thermodynamical, subjects have been treated at some length because of their affinity with the fundamental electromagnetic equations for vacuum. Returning now once more to Electromagnetism, we shall close this volume by dedicating the next chapter to Minkowski's equations for ponderable media.

NOTES TO CHAPTER IX.

Note 1 (to page 234). Let **i**, **j**, **k** be the *antecedents*, and $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ the *consequents** of the stress-dyadic f. Thus, if f, as in equation (5), is to be applied as a post-factor,

$$f=\mathbf{i}\;)\,\mathbf{f}_1+\mathbf{j}\;)\,\mathbf{f}_2+\mathbf{k}\;)\,\mathbf{f}_3. \qquad (a)$$

This means that $\mathbf{i}f=(\mathbf{ii})\mathbf{f}_1=\mathbf{f}_1$, etc., and in general,

$$\begin{aligned}\mathbf{n}f&=(\mathbf{ni})\mathbf{f}_1+(\mathbf{nj})\mathbf{f}_2+(\mathbf{nk})\mathbf{f}_3\\&=n_1\mathbf{f}_1+n_2\mathbf{f}_2+n_3\mathbf{f}_3,\end{aligned}$$

which is equal to $\mathbf{f}_n$, as it should be. Similarly, writing instead of **n** the Hamiltonian ∇,

$$\begin{aligned}\nabla f&=(\nabla\mathbf{i})\mathbf{f}_1+(\nabla\mathbf{j})\mathbf{f}_2+(\nabla\mathbf{k})\mathbf{f}_3\\&=\frac{\partial\mathbf{f}_1}{\partial x}+\frac{\partial\mathbf{f}_2}{\partial y}+\frac{\partial\mathbf{f}_3}{\partial z}.\end{aligned} \qquad \text{Q.E.D.}$$

Using the notation of Gibbs, *Scientific Papers*, Vol. II. p. 76, we should write $f=\mathbf{if}_1+\mathbf{jf}_2+\mathbf{kf}_3$, and

$$\frac{\partial\mathbf{f}_1}{\partial x}+\frac{\partial\mathbf{f}_2}{\partial y}+\frac{\partial\mathbf{f}_3}{\partial z}=\nabla . f,$$

where the dot means scalar application of ∇. But since, in our case, the prescription of applying ∇ scalarly is already given by the open parentheses in the dyadic (a), we do not require the dot or any other symbol of scalar multiplication.

Note 2 (to page 238). Let h, as in **Note 6** to Chapter VIII., be Minkowski's alternating matrix equivalent to the electromagnetic bivector, *i.e.* let, according to (c), p. 230,

$$h=\begin{vmatrix}0, & M_3, & -M_2, & -\iota E_1\\ -M_3, & 0, & M_1, & -\iota E_2\\ M_2, & -M_1, & 0, & -\iota E_3\\ \iota E_1, & \iota E_2, & \iota E_3, & 0\end{vmatrix}. \qquad (a)$$

Multiply it into itself. Then the first constituent of the first row of the resulting matrix hh will be

$$(hh)_{11}=-M_2^2-M_3^2+E_1^2=-f_{11}+\lambda,$$

where f_{11} is the corresponding component of the Maxwellian stress and $\lambda=\frac{1}{2}(M^2-E^2)$ the electromagnetic Lagrangian function per unit volume. Similarly,

$$(hh)_{12}=-f_{12},\quad (hh)_{13}=-f_{13},\quad (hh)_{14}=-\frac{\iota}{c}\mathfrak{P}_1=-\iota c g_1,$$

$$(hh)_{22}=-f_{22}-\lambda,\ \text{etc.},\quad (hh)_{44}=u-\lambda,$$

* This is Gibbs' nomenclature.

where u is the density of electromagnetic energy and **g** that of momentum, as throughout the chapter. Thus

$$-(hh)=\begin{vmatrix} f_{11}+\lambda, & f_{12}, & f_{13}, & \iota c g_1 \\ f_{21}, & f_{22}+\lambda, & f_{23}, & \iota c g_2 \\ f_{31}, & f_{32}, & f_{33}+\lambda, & \iota c g_3 \\ \iota c g_1, & \iota c g_2, & \iota c g_3, & -u+\lambda \end{vmatrix}=\mathfrak{S}+\lambda,$$

where $\mathfrak{S}$ is the matrix defined by (11), p. 238, and λ is written for λ times the unit matrix of 4×4 constituents. The required connexion is, therefore,

$$\mathfrak{S}=-hh-\lambda. \tag{b}$$

It will be remembered that λ is one of the invariants of h. And, since $h'=\bar{A}hA$, the last equation gives at once

$$\mathfrak{S}'=\bar{A}\mathfrak{S}A,$$

in agreement with (13), p. 239. On the quaternionic scheme we have, instead of (b), the operator **R**[ ]**L** of an analogous and somewhat simpler structure.

CHAPTER X.

MINKOWSKIAN ELECTROMAGNETIC EQUATIONS FOR PONDERABLE MEDIA.

In Minkowski's notes, which after his death were worked out by Dr. M. Born,* the electromagnetic equations for moving bodies, satisfying rigorously the principle of relativity, are deduced in a very ingenious way from the fundamental equations of the electron theory. And since the electronic equations were previously known to be invariant with respect to the Lorentz transformation, and gave to the relativist his first standard magnitudes, such a deduction was certainly very desirable and interesting. In fact, it occupied Minkowski's thought vividly during his last days. But in his own paper of 1907, repeatedly quoted, Minkowski adopts a purely phenomenological method, and deduces the equations for moving bodies, now generally associated with his name, from Maxwell's equations for stationary media by subjecting them to a Lorentz transformation.

In the present chapter we shall avail ourselves of the latter method only, which, apart from other considerations, recommends itself by its mathematical simplicity. Readers, and especially those who desire to see the electron theory made the foundation of all electromagnetic science, are referred to Dr. Born's paper just quoted, where the resulting equations † are wholly identical with Minkowski's original equations to be given presently.

* *Fortschritte der math. Wiss. in Monographien*, edited by O. Blumenthal, Heft 1, Teubner, 1910, p. 58.

† *E.g.* the differential equations of the field and the connexions between the various vectors involving the specific 'constants' of the medium, but not the formulae concerning ponderomotive action. These, as far as I know, have not yet been worked out electronically, for moving bodies. Einstein and Laub give an electronic deduction of the ponderomotive forces upon *stationary* media in

We shall retain here the notation adopted in Chapter II., where Maxwell's equations for a perfect insulator are collected under (3), p. 26. In the more general case of a conducting body, we have to supplement the displacement current by the conduction current. The latter, reckoned per unit area, we shall denote by $\mathbf{I}'$, and the electrical conductivity by σ. Thus, Maxwell's equations, written for the system S', in which the ponderable body is *at rest*, will consist of the two groups:

$$\left.\begin{aligned} &\frac{\partial \mathfrak{E}'}{\partial t'} + \mathbf{I}' = c.\,\text{curl}'\,\mathbf{M}'\,;\ \text{div}'\,\mathfrak{E}' = \rho' \\ &\frac{\partial \mathfrak{M}'}{\partial t'} = -c.\,\text{curl}'\,\mathbf{E}'\,;\ \text{div}'\,\mathfrak{M}' = 0, \end{aligned}\right\} \qquad (1')$$

independent of the properties of the particular body, and

$$\mathfrak{E}' = K\mathbf{E}', \quad \mathfrak{M}' = \mu\mathbf{M}', \quad \mathbf{I}' = \sigma\mathbf{E}', \qquad (2')$$

containing its specific 'constants.' These, the permittivity, inductivity (or permeability) and conductivity, which hereafter will play the part of invariants,* may be either simple scalars (more generally, linear vector operators) if dispersion is disregarded, or otherwise compound differential operators. In the latter case (in which practically K alone is concerned) the operator K is to be expressly constructed so as to be invariant. Thus it may consist of derivations of any order with respect to the proper time of the body.

In what follows we shall limit ourselves to *isotropic* media, so that K, μ, σ will have at any rate a *scalar character*, being either scalar magnitudes or scalar operators involving differentiations.

Let now S be another system of reference (say, the earth), relative to which our ponderable medium, together with its rest-system S', moves with a uniform velocity $\mathbf{v}$. Assuming the rigorous validity of Maxwell's equations (1′) and (2′) in S', and subjecting them to the appropriate Lorentz transformation, we shall obtain two groups of equations for the S-standpoint. Call them (1) and (2). What properties are we to require from (1) and (2) in the name of the

Ann. der Physik., Vol. XXVI. 1908, p. 541. Their formulae coincide, in the case of non-magnetic bodies, with those given by Lorentz in his article in *Encykl. der math. Wissenschaften*, Vol. V_2. 1904, pp. 245-250.

* This means that if the body *is at rest* in any other legitimate system S'', the connexions $\mathfrak{E}'' = K\mathbf{E}''$, etc., hold again with the same K, μ, σ.

principle of relativity? In the previous case of vacuum, when there was nothing to be carried along with the observers, all legitimate systems, S, S', S'', ... were wholly equivalent to one another, and the relativistic requirement was simply invariance or preservation of form of the equations. The case before us is different. The ponderable dielectric, with its specific properties, is at rest in one system at a time, and moves relatively to all other systems. The rest-system, in our concrete case S', is an uniquely privileged framework. If, in other concrete cases, the body were fixed in S or in S'', and so on, we should have to require the non-dashed or the double-dashed equations to be of the same form as the above (1′) and (2′). But, S being a system, relative to which the body does move (uniformly), we have to require only that the groups of equations (1) and (2), which might both contain the velocity $\mathbf{v}$,* should be invariant with respect to the Lorentz transformation by means of which we pass from S to any other legitimate system. If this requirement were not fulfilled, Maxwell's equations could not be used for relativistic purposes at all. But, as a matter of fact, they stand this test completely.

It seemed advisable to dwell a little upon these explanations; firstly, to avoid possible misunderstanding, and secondly, because the procedure and the test here exemplified are of general importance. They are the same in every other case in which the relativistic equations to be constructed concern any phenomena in ponderable bodies.

In order to obtain the two groups of equations, numbered in anticipation (1) and (2), and to see at the same time their invariance, put

$$\mathbf{L}' = \mathfrak{M}' - \iota\mathbf{E}', \quad \mathbf{R}' = \mathfrak{M}' + \iota\mathbf{E}'$$

and

$$\mathfrak{L}' = \mathbf{M}' - \iota\mathfrak{E}', \quad \mathfrak{R}' = \mathbf{M}' + \iota\mathfrak{E}',$$

and similarly for the non-dashed letters. Further, introduce the quaternion

$$C' = \iota\rho' + \frac{\mathbf{I}'}{c},$$

and write, as throughout the book, $D' = \partial/\partial l' + \nabla'$, using this operator as both a prefactor and a postfactor, as explained on p. 223.

* Though, as a matter of fact, only one of them will do so.

Then the four equations (1′) will assume the quaternionic form

$$\left.\begin{aligned} D'\mathfrak{L}' - \mathfrak{R}'D' &= 2C' \\ D'\mathbf{L}' + \mathbf{R}'D' &= 0. \end{aligned}\right\} \qquad (1'a)$$

These are identical with the first and the second pairs of (1′) respectively.

Now, let $Q[\ \ ]Q$ be our usual transformer from S to S', and therefore $Q_c[\ \ ]Q_c$ the inverse transformer. Apply the latter to each of the equations (1′a) and insert $Q_cQ=1$ between D' and $\mathfrak{L}'$ (or $\mathbf{L}'$), and $Q_cQ=1$ between $\mathfrak{R}'$ (or $\mathbf{R}'$) and D', in very much the same way as on p. 208. Then the result will be

$$DQ\mathfrak{L}'Q_c - Q_c\mathfrak{R}'QD = 2Q_cC'Q,$$

and similarly for the second equation, *i.e.*

$$\left.\begin{aligned} D\mathfrak{L} - \mathfrak{R}D &= 2C \\ D\mathbf{L} + \mathbf{R}D &= 0, \end{aligned}\right\} \qquad (1a)$$

where $\mathfrak{L} = Q\mathfrak{L}'Q_c$, $\mathfrak{R} = Q_c\mathfrak{R}'Q$, and similarly for the other pair of bivectors, and $C = Q_cC'Q_c$. Conversely,

$$\mathfrak{L}' = Q_c\mathfrak{L}Q, \text{ etc.,} \quad C' = QCQ.$$

In short, $C = \iota\rho + \mathbf{I}/c$ is a *physical quaternion*, $\mathfrak{L}$ and $\mathbf{L}$ are *left-handed physical bivectors*, and $\mathfrak{R}$ and $\mathbf{R}$ *right-handed* ones.* C may be called the (macroscopic) current-quaternion, while the electromagnetic bivectors need no special names.

Now, the S-equations (1a) are precisely of the same form as those, (1′a), for the rest-system. And so they will be also for every other legitimate system of reference. The velocity of the body does not, in fact, enter into these differential equations at all. We can now pass from their quaternionic form (1a) to the vectorial one, and shall thus obtain the required first group of equations:

$$\frac{\partial\mathfrak{E}}{\partial t} + \mathbf{I} = c \,.\, \mathrm{curl}\,\mathbf{M}, \text{ etc.,} \qquad (1)$$

exactly as in (1′) without the dashes. At the same time we have proved their invariance with respect to the Lorentz transformation.

* It will be remembered that the latter property is a necessary consequence of the former. In fact, as was proved in **Note 5** to Chap. VIII., p. 229, if $\mathbf{A} - \iota\mathbf{B}$ is a left-handed, then $\mathbf{A} + \iota\mathbf{B}$ is always a right-handed bivector.

This property finds its immediate expression in the above quaternionic form (1a).*

Moreover, the stated transformational properties of the electromagnetic bivectors and of the current-quaternion lead at once to the second group of equations for the moving body, to be deduced from the Maxwellian connexions (2′). In fact, since both $\mathbf{L}=\mathfrak{M}-\iota\mathbf{E}$ and $\mathfrak{L}=\mathbf{M}-\iota\mathfrak{E}$ are left-handed bivectors,† we have in exactly the same way as on p. 210, writing again ϵ for the longitudinal stretcher of ratio $\gamma=(1-v^2/c^2)^{-\frac{1}{2}}$,

$$\mathfrak{M}=\gamma\left[\frac{1}{\epsilon}\mathfrak{M}'+\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}'\right];\quad \mathbf{M}=\gamma\left[\frac{1}{\epsilon}\mathbf{M}'+\frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{E}'\right]$$

$$\mathbf{E}=\gamma\left[\frac{1}{\epsilon}\mathbf{E}'-\frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}'\right];\quad \mathfrak{E}=\gamma\left[\frac{1}{\epsilon}\mathfrak{E}'-\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{M}'\right],$$

whence, by the first and second of the connexions (2′), and after an easy rearrangement of terms,

$$\mathfrak{E}+\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{M}=K[\mathbf{E}+\frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}]$$

$$\mathfrak{M}-\frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}=\mu\,[\mathbf{M}-\frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{E}].$$

Both of these relations, involving the substantial properties of the medium, contain its velocity. Again, since $C=\iota\rho+\mathbf{I}/c$ is a physical quaternion, we have, by (1′b) of Chap. V., p. 125,

$$\mathbf{I}=\epsilon\mathbf{I}'+\gamma\rho'\mathbf{v}$$

and

$$\rho'=\gamma\,[\rho-\frac{1}{c^2}\,(\mathbf{I}\mathbf{v})],$$

whence, by the last of (2′),

$$\mathbf{I}=\sigma\epsilon\mathbf{E}'+\gamma^2[\rho-\frac{1}{c^2}(\mathbf{I}\mathbf{v})]\,\mathbf{v}.$$

* Or in Minkowski's matrix form. This consists of the two equations

$$\mathrm{lor}\,h=-s,\quad \mathrm{lor}\,H^*=0,$$

in which s is the matrix-equivalent of C, h and H the alternating matrices corresponding to the bivectors $\mathfrak{L}$ and $\mathbf{L}$ respectively, and H^* the dual of H.

† Notice, in passing, that this being the case, $\mathfrak{L}^2$ and $\mathbf{L}^2$ are complex invariants. These split into the four real invariants,

$$\mathfrak{M}^2-E^2,\quad (\mathfrak{M}\mathbf{E}),\quad M^2-\mathfrak{E}^2,\quad (\mathbf{M}\mathfrak{E}).$$

But $\frac{1}{\gamma}\mathbf{E}' = \frac{1}{\epsilon}\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}$. Hence, after a slight rearrangement of terms,

$$\mathbf{I} - \rho\mathbf{v} = \mathfrak{I} = \sigma\gamma\frac{1}{\epsilon^2}[\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}].$$

Thus $\mathbf{I}$ appears as the sum of the *convection current* $\rho\mathbf{v}$ and the *conduction current*, for which we have written $\mathfrak{I}$, the latter being proportional to the conductivity.*

Using the convenient abbreviations

$$\left.\begin{aligned} \mathbf{E}^{\times} &= \mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}, \quad \mathfrak{E}^{\times} = \mathfrak{E} + \frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{M} \\ \mathbf{M}^{\times} &= \mathbf{M} - \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{E}, \quad \mathfrak{M}^{\times} = \mathfrak{M} - \frac{1}{c}\mathrm{V}\mathbf{v}\mathbf{E}, \end{aligned}\right\} \qquad (\mathrm{A})$$

and gathering together the above results, we obtain the required second group of equations, valid from the standpoint of the system S,

$$\left.\begin{aligned} \mathfrak{E}^{\times} = K\mathbf{E}^{\times}, \quad \mathfrak{M}^{\times} = \mu\mathbf{M}^{\times} \\ \mathbf{I} - \rho\mathbf{v} = \mathfrak{I} = \sigma\gamma\frac{1}{\epsilon^2}\mathbf{E}^{\times}. \end{aligned}\right\} \qquad (2)$$

These three connexions involve the velocity of the ponderable medium relative to that system. It remains only to prove that they are invariant with respect to the Lorentz transformation. Now, introducing the velocity-quaternion

$$Y = \gamma[\iota c + \mathbf{v}],$$

we have, identically,

$$\left.\begin{aligned} \eta \equiv \frac{1}{2c}[Y\mathbf{L} - \mathbf{R}Y] &= \gamma\left[\frac{\iota}{c}(\mathbf{E}^{\times}\mathbf{v}) + \mathbf{E}^{\times}\right], \\ \frac{1}{2c\iota}[Y\mathbf{L} + \mathbf{R}Y] &= \gamma\left[\frac{\iota}{c}(\mathfrak{M}^{\times}\mathbf{v}) + \mathfrak{M}^{\times}\right], \\ \frac{1}{2c}[Y\mathfrak{L} - \mathfrak{R}Y] &= \gamma\left[\frac{\iota}{c}(\mathfrak{E}^{\times}\mathbf{v}) + \mathfrak{E}^{\times}\right], \\ \zeta \equiv \frac{1}{2c\iota}[Y\mathfrak{L} + \mathfrak{R}Y] &= \gamma\left[\frac{\iota}{c}(\mathbf{M}^{\times}\mathbf{v}) + \mathbf{M}^{\times}\right], \end{aligned}\right\} \qquad (\mathrm{B})$$

* Adding the displacement current, we should have

$$\partial\mathfrak{E}/\partial t + \rho\mathbf{v} + \mathfrak{I},$$

the 'total' current. This is, by the first of (1), always solenoidal.

and each of these expressions* is a *physical quaternion*, $\simeq q$. Moreover, starting from the current-quaternion C and its conjugate C_c, we easily obtain the identical equation

$$\frac{c}{2}\left[C+\frac{1}{c^2}YC_cY\right]=\epsilon^2\mathbf{I}-\rho\gamma^2\mathbf{v}+\frac{\iota}{c}[(\mathbf{Iv})-\rho v^2]\gamma^2,$$

of which the left-hand side is, obviously, again a physical quaternion. So also is its right-hand side, which, by the third of (2), is equal to $\sigma\eta$. Using, therefore, the above identities we can write the whole of (2), in terms of physical quaternions alone,

$$\left.\begin{aligned} Y\mathfrak{L}-\mathfrak{R}Y&=K[Y\mathbf{L}-\mathbf{R}Y]\\ Y\mathbf{L}+\mathbf{R}Y&=\mu[Y\mathfrak{L}+\mathfrak{R}Y]\\ C+\frac{1}{c^2}YC_cY&=\frac{\sigma}{c^2}[Y\mathbf{L}-\mathbf{R}Y]. \end{aligned}\right\} \qquad (2a)$$

This proves the invariance of the relations (2) with respect to the Lorentz transformation.† Thus the whole of equations (1) and (2) satisfy the principle of relativity. Q.E.D.

It is worth noticing here that the world-vector corresponding to the quaternion

$$\tfrac{1}{2}\left[C+\frac{1}{c^2}YC_cY\right]$$

is the part of the four-current C *normal* to the four-velocity Y. Generally, for any pair of physical quaternions a, b, the expression

$$\tfrac{1}{2}a-\frac{ba_cb}{2(\mathrm{T}b)^2}$$

represents that part of the four-vector corresponding to a, which is normal to the four-vector b (**Note 1**). The above statement is deduced from this, remembering that $\mathrm{T}Y=\iota c$.

* Of which the first and the last, denoted for subsequent reference by η and ζ, are the quaternionic equivalents of Minkowski's world-vectors of the first kind Φ and Ψ, called by him *elektrische Ruh-Kraft* and *magnetische Ruh-Kraft* respectively. Cf. his *Grundgleichungen*, pp. 33-34.

† Minkowski's matrix-form of the above relations is

$$Yh=KYH\,;\quad YH^*=\mu Yh^*\,;\quad s+Y\bar{s}Y=-\sigma YH,$$

where Y is the matrix corresponding to the quaternion Y, and the remaining symbols are as in footnote on p. 264. In these formulae we have put, after Minkowski, $c=1$.

In the course of the above calculations we came across the formula $\rho'/\gamma = \rho - (\mathbf{Iv})/c^2$. Its inversion will be

$$\rho = \gamma\left[\rho' + \frac{1}{c^2}(\mathbf{I'v})\right].$$

Substituting here $(\mathbf{I'v}) = \gamma(\mathbf{Iv}) - \gamma\rho v^2$ and remembering that $\mathbf{I} = \mathfrak{I} + \rho\mathbf{v}$, we obtain the interesting relation

$$\rho = \gamma\rho' + \frac{\gamma^2}{c^2}(\mathfrak{I}\mathbf{v}), \tag{3}$$

about which a few words will be said later on. To resume the above results:

The equations for a moving isotropic* conducting dielectric, obtained from Maxwell's equations for stationary media, are invariant with respect to the Lorentz transformation. They consist 1° of a set of differential equations not containing the velocity of motion at all, and 2° of a set of relations concerning the substantial properties of the medium and involving its velocity $\mathbf{v}$ relative to the observing system. The quaternionic form of these two sets of equations is given in (1*a*) and (2*a*), where $\mathfrak{L}$, $\mathbf{L}$ are left-handed and $\mathfrak{R}$, $\mathbf{R}$ right-handed physical bivectors, and C a physical quaternion, $\simeq q$. The vector form of the first set is

$$\left.\begin{aligned} &\frac{\partial \mathfrak{E}}{\partial t} + \mathbf{I} = c\,.\,\mathrm{curl}\,\mathbf{M}; \quad \mathrm{div}\,\mathfrak{E} = \rho \\ &\frac{\partial \mathfrak{M}}{\partial t} = -c\,.\,\mathrm{curl}\,\mathbf{E}; \quad \mathrm{div}\,\mathfrak{M} = 0 \end{aligned}\right\} \tag{1}$$

and that of the second set

$$\left.\begin{aligned} &\mathfrak{E}^\times = K\mathbf{E}^\times, \quad \mathfrak{M} = \mu\mathbf{M}^\times \\ &\mathbf{I} - \rho\mathbf{v} = \mathfrak{I} = \sigma\gamma\epsilon^{-2}\mathbf{E}^\times, \end{aligned}\right\} \tag{2}$$

where $\mathbf{E}^\times$ stands for $\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{v}\mathfrak{M}$, etc., as in (A), and K, μ, σ for the permittivity, inductivity and conductivity of the body, as originally defined from the standpoint of the rest-system.

* If K, etc., were vector operators, the passage from (2) to (2*a*), *via* (B), would not be legitimate. In fact, $(\mathfrak{E}^\times\mathbf{v})$ would then be equal to $(K\mathbf{E}^\times . \mathbf{v})$, which has nothing to do with $K(\mathbf{E}^\times\mathbf{v})$, the former expression being a scalar and the latter an operator. It is for this reason only that we have limited ourselves to *isotropic* bodies. The case of anisotropy has not, to my knowledge, yet been treated, and may be left for the reader's own investigation.

These are *Minkowski's equations.* They were first given in his fundamental paper of 1907, in both their vectorial and matricular forms already quoted. We may notice here that Minkowski himself assumed that Maxwell's equations (1′) and (2′) are valid (in the corresponding instantaneous rest-system S') at each point of the material body, whatever the state of motion around that point, just as if the whole body were fixed in S'. It is this that he calls his 'first axiom' (*loc. cit.*, § 8). Such being Minkowski's starting-point, he asserts, consequently, the validity of the resulting equations (1) and (2) for each element of a material medium moving in an arbitrary manner with respect to the framework S, in short, for $\mathbf{v}$ varying in both space and time. His only restriction is that $v<c$. Now, it is not unlikely that the first set of Minkowski's equations can claim such a general validity. (Notice that these are, properly speaking, two equations for five vectors, otherwise yet unconnected.) But the case is different when the first set is supplemented by the second. For, apart from other reasons, if we pass to $K=\mu=1$ and $\sigma=0$, the whole of equations (1), (2) reduce, as will be seen presently, to the vacuum-equations, and the acceptance of the latter for frameworks whose relative motion is variable, would require a thorough reconstruction of the principle of relativity underlying the whole theory. Retaining, therefore, this principle, we can consider Minkowski's equations as *rigorously* valid only for *uniform motion.* Accordingly our $\mathbf{v}$ has been treated from the outset as a constant vector and Y as a constant velocity-quaternion belonging to the body as a whole. Of course, as an *approximation* of more than sufficient accuracy, the equations (1) and (2) can well be used for velocities experiencing all such time- and space-variations as are practically realizable. Thus, for instance, they can safely be applied to bodies kept rotating, as in the case of Wilson's experiment; the unequal FitzGerald-Lorentz contraction and the ensuing stress with its influence upon K, etc., being of the order of β^2.

The comparison of the equations (1), (2) with those of Hertz-Heaviside, Lorentz and Cohn, none of which satisfy rigorously the principle of relativity, must be left to the reader. It is given at sufficient length in Minkowski's paper. As to Hertz-Heaviside's equations for moving bodies, we have already seen that they are not even a first-order approximation to the observed state of things, giving a full, instead of the Fresnelian, drag. In fact, Hertz-Heaviside's equations are, by their very construction

invariant with respect to the Newtonian, and not to the Lorentz transformation.

Let us now stop a while at Minkowski's equations in order to learn some of their properties.

In the first place, if $K=\mu=1$ and $\sigma=0$, then Minkowski's equations reduce at once to the fundamental or *the vacuum-equations*. In fact, in this limiting case we have, by the third of (2), $\mathbf{I}=\rho\mathbf{v}$, and if $\rho'=0$, also $\rho=0$, by (3). Again, by the first and second of (2), $\mathfrak{E}^\times=\mathbf{E}^\times$ and $\mathfrak{M}^\times=\mathbf{M}^\times$, *i.e.*

$$\mathfrak{E}-\mathbf{E}=\frac{1}{c}\mathrm{V}\mathbf{v}[\mathfrak{M}-\mathbf{M}],$$

$$\mathfrak{M}-\mathbf{M}=-\frac{1}{c}\mathrm{V}\mathbf{v}[\mathfrak{E}-\mathbf{E}],$$

whence, by elimination,

$$\mathfrak{E}-\mathbf{E}=\beta^2(\mathfrak{E}-\mathbf{E}),$$

and since $\beta\neq 1$, $\mathfrak{E}=\mathbf{E}$, and similarly, $\mathfrak{M}=\mathbf{M}$. Q.E.D. The same result may be obtained from the quaternionic form (2*a*). In the present case $\mathfrak{L}=\mathbf{L}$ becomes identical with the electromagnetic bivector of the preceding chapters. And since at the same time $\mathfrak{R}=\mathbf{R}$, the sum of the equations (1*a*) gives at once $D\mathbf{L}=C$. Properly speaking, to obtain $K=\mu=1$, $\sigma=0$, we have (on the electro-atomistic doctrine) to consider a region outside the electrons, or at least outside electronic assemblages crowded within atomic regions. Then $\rho=0$, $D\mathbf{L}=0$, and here the macroscopic bivector coincides with our previous microscopic $\mathbf{L}$. Thus the announced reduction becomes complete.

As regards the meaning of the vector $\mathbf{I}$, we have already remarked that it is the sum of the convection- and the conduction-current. In virtue of the properties of the stretcher ϵ, the longitudinal component of the latter current will be

$$\mathfrak{I}_1=\frac{\sigma}{\gamma}E_1^\times=\frac{\sigma}{\gamma}E_1,$$

and the transversal ones

$$\mathfrak{I}_2=\sigma\gamma E_2^\times,\quad \mathfrak{I}_3=\sigma\gamma E_3^\times.$$

This is in explanation of the short form of the third of (2), which may be looked at as the expression of *Ohm's law*. If, for instance, $\epsilon^{-2}\mathbf{E}^\times$ is considered as the resultant E.M.F. per unit length, then $1/\sigma\gamma$ will be the specific resistance for the S-standpoint. This is one simple

way of splitting the conduction current into factors. But since, thus far, the only requirement is that 'resistance' should reduce to $1/\sigma$ for $v=0$, we may equally well give the name of 'electromotive force' to the line-integral of the vector $\mathbf{E}^\times$ itself; then we shall have the specific resistance-operator $\epsilon^2/\sigma\gamma$, instead of an ordinary scalar. If second-order terms are neglected, the distinction disappears. The conduction current may then be written, with more than sufficient approximation,

$$\mathfrak{J} \doteqdot \sigma\mathbf{E}^\times.$$

We will not stop here to discuss the nomenclature proposed by various authors for $\mathbf{E}^\times$ and its magnetic companion. It seems advisable to leave them for the time being without any names.

The integral properties of $\mathbf{E}^\times$ and $\mathbf{M}^\times$, in relation to $\mathfrak{M}$, etc., may at once be put into a form with which the reader has become familiar in Chapter II. In fact, by (A) and (1), we have

$$-c\,.\,\mathrm{curl}\,\mathbf{E}^\times = -c\,.\,\mathrm{curl}\,\mathbf{E} - \mathrm{curl}\,\mathrm{V}\mathbf{v}\mathfrak{M}$$
$$= \frac{\partial\mathfrak{M}}{\partial t} + \mathbf{v}\,\mathrm{div}\,\mathfrak{M} + \mathrm{curl}\,\mathrm{V}\mathfrak{M}\mathbf{v},$$

and this is precisely what in **Note 2** to Chap. II. has been called

current $(\mathfrak{M})$.

That is to say, if $d\sigma$ be a surface element composed always of the same particles of the body, and $\mathbf{n}$ the normal of $d\sigma$, we have

$$c(\mathbf{n}\,.\,\mathrm{curl}\,\mathbf{E}^\times) = -\frac{d}{dt}(\mathfrak{M}\mathbf{n}\,d\sigma).$$

Similarly,

$$c(\mathbf{n}\,.\,\mathrm{curl}\,\mathbf{M}^\times) = \frac{d}{dt}(\mathfrak{E}\mathbf{n}\,d\sigma) + (\mathfrak{J}\mathbf{n}).$$

Recurring, therefore, to Stokes' theorem, we have for any surface σ, which together with its bounding circuit s *is carried along with the body*,

$$\int_{(s)}(\mathbf{M}^\times d\mathbf{s}) = \frac{1}{c}\int(\mathfrak{J}\mathbf{n})\,d\sigma + \frac{1}{c}\,\frac{d}{dt}\int(\mathfrak{E}\mathbf{n})\,d\sigma, \qquad (4)$$

$$\int_{(s)}(\mathbf{E}^\times d\mathbf{s}) = -\frac{1}{c}\,\frac{d}{dt}\int(\mathfrak{M}\mathbf{n})\,d\sigma. \qquad (5)$$

These are the required formulae. Returning to p. 23 (where Maxwell's law I. is to be supplemented by the conduction current)

and to p. 30, the reader will find this integral form of equations most suitable for a direct comparison of Hertz's theory with that of Minkowski. Instead of Hertz's $\mathbf{E}$, $\mathbf{M}$ we have here $\mathbf{E}^\times$, $\mathbf{M}^\times$, and instead of his $\mathfrak{E} = K\mathbf{E}$, etc., the Minkowskian relations (2), involving the velocity of the medium relative to the observing system.

Applying (4) to a pair of surfaces bounded by one and the same circuit s, as on p. 25, we obtain the familiar equation,

$$\frac{de}{dt} = -\int (\mathfrak{I}\mathbf{n})\, d\sigma, \tag{6}$$

where e is the total charge of any portion of the medium enclosed completely by the surface σ, whose outward normal is $\mathbf{n}$. If the bounding surface is entirely composed of lines of conduction-current, then the charge remains constant. The same result follows, of course, from the first pair of the differential equations (1), with $\mathbf{I} = \rho\mathbf{v} + \mathfrak{I}$. And since these are independent of the Minkowskian connexions, involving the substantial properties of the medium, there is no wonder that the equation of continuity reappears in its familiar form.

The above equations (4) and (5) lead at once to a pair of what are usually called the boundary conditions. The other pair follows directly from $\operatorname{div}\mathfrak{E} = \rho$ and $\operatorname{div}\mathfrak{M} = 0$. In fact, let Σ be, in Hadamard's phraseology, a *stationary* surface of discontinuity,* *i.e.* permanently affecting the same material particles, such as the surface of contact of two different media. And let us require that $\mathfrak{I}$ and the *individual* time-rate of change of $\mathfrak{E}$ and $\mathfrak{M}$ should be *finite.* This condition, to be fulfilled at any point of Σ and elsewhere, is necessary to prevent $\mathfrak{E}$, $\mathfrak{M}$ mounting up to infinite values at any point of the medium.† Under these assumptions apply (4) and (5), in the usual way, to an infinitesimal rectangle, with its shorter sides normal to Σ. Then the result will be that the tangential components of $\mathbf{E}^\times$ and $\mathbf{M}^\times$ must be continuous. The two remaining conditions are as in the older theory. They follow at once from the divergence-formulae, and require the normal component of $\mathfrak{M}$ to be continuous, and the

*To be carefully distinguished from a *wave* of discontinuity, which is propagated in the material medium. The reader unfamiliar with this subject is referred to the author's *Vectorial Mechanics*, pp. 128 *et seq.*

† While it is not necessary at all for a wave, whose singularities do not remain at the same particles, but pass by and are transferred to others and others.

jump of the normal component of $\mathfrak{E}$ to be equal to the surface-density of charge. Thus, if there is no such charge, we have the following *boundary conditions*:

$$\begin{gathered}(\mathfrak{M}\mathbf{n}) \text{ and } (\mathfrak{E}\mathbf{n}) \textit{ continuous}, \\ V\mathbf{n}V\mathbf{E}^{\times}\mathbf{n} \text{ and } V\mathbf{n}V\mathbf{M}^{\times}\mathbf{n} \textit{ continuous},\end{gathered} \tag{7}$$

where $\mathbf{n}$ is normal to the boundary. The latter pair of expressions gives the tangential *parts* of the vectors, *i.e.* in both size and direction.

Next, as regards the formula (3) for the density of charge, which is a consequence of the nature of C as a physical quaternion. Suppose, first, that there is no conductivity. Then

$$\rho = \gamma\rho',$$

just as for the microscopic density of charge, whence, for any portion of the body,

$$e = \int \rho\, dS = \int \rho' dS' = e',$$

which means relativistic invariance of macroscopic charge. This property then continues to hold for a moving body, provided that it is a perfect *insulator.*

On the other hand, suppose that the body is *conductive*, but that there is no rest-charge ($\rho' = 0$). Then there will be for the S-observers an apparent charge of density

$$\rho = \frac{\gamma^2}{c^2}(\mathfrak{I}\mathbf{v}). \tag{8}$$

The history of this *conduction charge*, or *compensation charge*, as it previously has been called, can be traced back as far as 1880, in which year it was deduced by Budde (*Wied. Ann.*, Vol. X. p. 553) from Clausius' fundamental law of electrodynamics. Budde, whose formula differed from the above one by containing unity instead of γ^2, was able to defend Clausius' law from a serious attack by showing that this charge accounted for the non-existence of an action between a current circuit and a charged body sharing in the earth's motion. Hence the name of 'compensation charge.' In 1895 Lorentz, by averaging his electronic equations, obtained for the density of this charge a formula which was wholly identical with (8). See § 25 of his *Essay.* A careful comparison of the

two ways leading to one and the same result will be found useful, and the electronic interpretation of a formula which here appears as a relativistic consequence of Maxwell's equations will not be lacking in interest. But even apart from electro-atomistic concepts the reader will not fail to see that if the densities of positive electricity, flowing one way, and negative flowing the other way, cancel one another for an observer attached to the conducting body, then the corresponding values ρ_+ and ρ_-, as estimated from any other (S-) point of view, will in general not annul themselves. They will do so only when the current has no longitudinal component. There is no difficulty in working out the quantitative details of such a reasoning, and thus re-obtaining the above formula.

Next, as regards the dragging of waves. We know already from Chapter VI. that, whatever the value $\mathfrak{v}'$ of the velocity of propagation in the rest-system, its S-value $\mathfrak{v}$ will follow by the addition theorem of velocities, and will give, therefore, the Fresnelian coefficient. And that Einstein's theorem is in fact applicable to the present case, can be concluded from the manner in which the equations (1), (2) have been obtained from those, (1′), (2′), holding in S'. Thus we know beforehand that Minkowski's equations will lead to the correct Fresnelian value of the dragging coefficient. And this expectation is readily confirmed on performing the explicit calculation. Cf. **Note 2.**

Finally, let us remark that Minkowski's electromagnetic equations account fully for the well-known results of Rowland's, Wilson's, Röntgen's and Eichenwald's experiments. We cannot enter here upon the corresponding details, and must confine ourselves to short indications concerning each of these famous experiments. The magnetic effect of the *convection current*, first proved experimentally by Rowland, and confirmed by other physicists,* is directly expressed by the term $\rho\mathbf{v}$, which together with the conduction current makes up $\mathbf{I}$, and thus equally with that current contributes to the magnetic field. It is scarcely necessary to say that the Rowland effect was equally well expressed by the Hertz-Heaviside equations. The result of *Wilson's experiments* on the

* H. A. Rowland, *Amer. Journ. of Science*, Vol. XV. 1878, p. 30. H. A. Rowland and C. T. Hutchinson, *Phil. Mag.*, Vol. XXVII. 1889, p. 445. H. Pender, *Phil. Mag.*, Vol. II. 1901, p. 179. E. P. Adams, *ibidem*, p. 285. H. Pender and V. Crémieu, *Comptes rendus*, Vol. CXXXVI. 1903, pp. 548, 955. A. Eichenwald, *Ann. der Physik*, Vol. XI. 1903, p. 1.

electric effect of rotating a dielectric between the connected plates of a condenser in a magnetic field M consisted in each of the plates being found charged to a surface-density

$$(K-1)\beta M \qquad \text{(Wilson)}$$

of opposite signs.* In the theoretical treatment of the problem uniform translation (of each element) can, with sufficient accuracy, be substituted for the actual spin, and the state being supposed stationary (and $\sigma=0$, $\rho=0$), Minkowski's differential equations reduce to $\operatorname{curl}\mathbf{E}=0$, etc. Using these, with the appropriate boundary-conditions, and the first pair of (2), Einstein and Laub † deduce, for the surface-density in question, the value

$$(K\mu-1)\beta M, \qquad \text{(Mnk)}$$

with the correct sign for each plate. The authors observe that Lorentz's theory would give, instead of this,

$$(K-1)\beta\mu M. \qquad \text{(Lor)}$$

Since in Wilson's case μ was $=1$, both of these theoretical formulae coincide with his experimental result. If a dielectric of considerable inductivity were available, experiment would readily decide in favour of the former or the latter theory. As to Hertz-Heaviside's theory, it would give for the Wilson-effect

$$K\mu\beta M, \qquad \text{(HH)}$$

i.e. practically $K\beta M$, which is equally contradicted by Wilson's and by Blondlot's results. This disagreement, even in the case of a first-order effect, might have been expected, in view of the fact that Hertz-Heaviside's equations give a full instead of a Fresnelian drag. Lastly, as regards the experiments on the magnetic effect of moving polarized dielectrics, which were first carried out by Röntgen and more recently with increased accuracy by Eichenwald,‡ it will be enough to write down the expression of what is generally called

* H. A. Wilson, *Phil. Trans.*, Vol. CCIV. A, 1904, p. 121. Wilson's positive result agrees with the absence of any such effect stated previously by R. Blondlot, *Comptes rendus*, Vol. CXXXIII. 1901, p. 778, in the case of *air* as dielectric, for which K differs but little from unity.

† A. Einstein and J. Laub, *Ann. der Physik*, XXVI. 1908, p. 532.

‡ W. C. Röntgen, *Berl. Sitzungsberichte*, 1885, p. 195; *Wied. Ann.*, Vol. XXXV. 1888, p. 264, and Vol. XL. 1890, p. 93. A. Eichenwald, *Ann. der Physik*, Vol. XI. 1903, p. 421.

the *Röntgen-current.* In fact, if we limit ourselves to homogeneous media, the experimental results may be concisely stated by saying that the observed value of the Röntgen-current is

$$(K-1)\,\mathrm{curl}\,\mathrm{V}\mathbf{E}\mathbf{v}. \qquad \text{(Exper.)}$$

Now, according to the Hertz-Heaviside equations (p. 31), this current would be

$$K.\,\mathrm{curl}\,\mathrm{V}\mathbf{E}\mathbf{v}, \qquad \text{(HH)}$$

so that the disagreement is exactly of the same kind as for the Wilson-effect. On the other hand, Minkowski's equations, with $\mu=1$, give for the Röntgen-current the rigorous value

$$\mathrm{curl}\,\mathrm{V}[\mathfrak{E}-\mathbf{E}]\mathbf{v}, \qquad \text{(Mnk)}$$

where, by the first of (2) and by (A),

$$\mathfrak{E}-\mathbf{E}=(K-1)\mathbf{E}+\frac{1}{c}\mathrm{V}\mathbf{v}[K\mathfrak{M}-\mathbf{M}].$$

Thus the first-order term of the Minkowskian expression represents correctly the observed facts. The second-order terms are, of course, for the time being far too small to be detected. The Minkowskian value of the Röntgen-current follows also from a later form of Lorentz's equations deduced (1902) from the electron theory.* In what consists the violation done by these last equations to the principle of relativity may be seen from Minkowski's paper. There the reader will find also the appropriate coordination of the field-vectors involved in the various theories.

So much as regards the electromagnetic equations for moving bodies, contained in (1) and (2). Now for the dynamical part of the subject. Before proceeding to a relativistic construction of the formulae for the ponderomotive force and the associated physical magnitudes, some preliminary remarks seem indispensable. These will concern the requirements to be postulated in addition to those dictated by the principle of relativity itself. The choice of such supplementary requirements or postulates is free, within fairly wide limits. We shall select those which seem to offer the advantage of possible simplicity and which will lead to results but slightly different from the ponderomotive formulae originally proposed by Minkowski.

* *Amsterdam Proceedings*, 1902-1903, p. 254. See Lorentz's article in *Encykl. der math. Wiss.*, Vol. V_2. pp. 208-211, and in particular formula (XXVII.), in which $\mathfrak{P}=\mathfrak{D}-\mathfrak{E}$ corresponds to the above $\mathfrak{E}-\mathbf{E}$.

Let $\mathbf{P}$ be *the ponderomotive force* due to the electromagnetic field, per unit volume of the medium, and, therefore, $(\mathbf{P}\mathbf{v})$ its activity. Further, let J be *Joule's heat*, or the Joulean waste, per unit time and unit volume, and F the force-quaternion, *i.e.*, according to what has been said in the last chapter,

$$F=\frac{\iota}{c}[(\mathbf{P}\mathbf{v})+J]+\mathbf{P}. \tag{9}$$

Let u be the density of electromagnetic energy, $\mathbf{g}$ that of electromagnetic momentum, and finally f and $\mathfrak{P}$ the ('absolute,' not relative) stress-operator and flux of energy, as defined in the usual way with respect to the observing system S. With this meaning of the symbols, let our requirements be as follows:

1°. *F, a physical quaternion,*

$$F\equiv\frac{\iota}{c}[(\mathbf{P}\mathbf{v})+J]+\mathbf{P}\simeq q. \tag{α}$$

2°. *Principle of momentum*, to call it by its usual short name, that is to say,

$$\mathbf{P}=-\nabla f-\frac{\partial\mathbf{g}}{\partial t}, \tag{β}$$

where ∇f stands for $\partial\mathbf{f}_1/\partial x+\partial\mathbf{f}_2/\partial y+\partial\mathbf{f}_3/\partial z$.

3°. *Principle of conservation of energy*, *i.e.*

$$(\mathbf{P}\mathbf{v})+J=-\frac{\partial u}{\partial t}-\operatorname{div}\mathfrak{P}, \tag{γ}$$

where $\mathfrak{P}$ has, *thus far*, *nothing to do with the momentum.*

It is needless to add that, besides fulfilling these explicit requirements, the resulting formulae have to agree with experience, as far as it goes, and to reduce, for $K=\mu=1$, $\sigma=0$, to the previous vacuum-formulae, as, in fact, they will.

We have seen in the preceding chapter that there is at the present time a strong tendency to universalize the simple relation of equality holding between $\mathbf{g}$ and $\mathfrak{P}/c^2$ in the ideal limiting case of a vacuum.*

*This tendency was initiated by Planck's paper (*Phys. Zeitschrift*, Vol. IX. 1908, p. 828) on the principle of action and reaction. M. Abraham uses the equality $c^2\mathbf{g}=\mathfrak{P}$ throughout his papers (quoted on p. 240), putting it at the base of his electrodynamics of moving bodies, which is also adopted in Laue's *Relativitätsprinzip*. That equality is called by Laue 'the theorem of the inertia of energy,' and plays in his book the part of an universally valid relation. But his own way of introducing this 'theorem' (p. 164 of the 2nd ed.) will show best how vague are the reasons for accepting it without limitation.

But, as far as I can see, there is nothing to compel us to such a generalization. If it is assumed that the matrix embodying the stress, momentum, etc., should be symmetrical, then, of course, the equality under consideration follows from (β) and (γ). But nothing prevents us from abandoning, at least in the case of ponderable media, that assumption of symmetry.* We shall see that in doing so we need not even give up the formulae (14) or (14a) of Chap. IX., which have led to so many far-reaching consequences. These formulae will continue to hold within wide limits, although the more general formula (10) of that chapter will have to be modified. Thus, there will still be 'inertia of energy,' with its manifold corollaries.

So much to justify the abandoning of the assumption of universal proportionality of momentum and energy-flux.

Returning to our above requirements, let us, first of all, observe that, with the given meaning of F, assumptions (β) and (γ) may be condensed into

$$F = -\operatorname{lor} \mathfrak{S}, \tag{10}$$

where

$$\mathfrak{S} = \begin{vmatrix} f, & \frac{\iota}{c}\mathfrak{p} \\ \iota c\mathbf{g}, & -u \end{vmatrix} \tag{11}$$

or, written out fully,

$$\mathfrak{S} = \begin{vmatrix} f_{11}, & f_{12}, & f_{13}, & \frac{\iota}{c}\mathfrak{p}_1 \\ f_{21}, & f_{22}, & f_{23}, & \frac{\iota}{c}\mathfrak{p}_2 \\ f_{31}, & f_{32}, & f_{33}, & \frac{\iota}{c}\mathfrak{p}_3 \\ \iota c g_1, & \iota c g_2, & \iota c g_3, & -u \end{vmatrix}. \tag{11a}$$

*In Sommerfeld's four-dimensional algebra (*loc. cit.*), the symmetrical world-tensor, corresponding to such a matrix, is generated by what he calls 'a complete multiplication' of a six-vector into itself. But why not multiply two *different* six-vectors 'completely' into one another? Such a procedure is exemplified, in matrix-form, in Minkowski's paper. But, apart from the process of generation, any given matrix of 4×4 constituents can be used for relativistic purposes, provided that its product into a four-vector (matrix) gives again a four-vector.

Here, in general, $f_{\iota\kappa} \neq f_{\kappa\iota}$, so that the matrix lacks symmetry altogether.

Next, to satisfy (α), we have to write, for any pair of legitimate frameworks of reference S and S', as on p. 239,

$$\mathfrak{S} = A\mathfrak{S}'\bar{A}, \tag{12}$$

where $A, \bar{A}$ are as before. This fixes the transformational properties of the stress, momentum, etc., quite independently of the electromagnetic expressions they will hereafter receive. Developing (12), we have the following table of Cartesian formulae, which take the place of (10a), p. 237, and which, though not needed for our electrodynamical investigation, are here given because of their bearing upon the subjects treated in the preceding chapter:

$$\left.\begin{array}{l}
f_{11}=\gamma^2[f_{11}'+\beta^2u'+\frac{\beta}{c}(\mathfrak{P}_1'+c^2g_1')]\,;\;\; f_{22}=f_{22}'\,;\;\; f_{33}=f_{33}' \\
f_{23}=f_{23}'\,;\;\; f_{31}=\gamma(f_{31}'+\frac{\beta}{c}\mathfrak{P}_3')\,;\;\; f_{12}=\gamma(f_{12}'+vg_2') \\
f_{32}=f_{32}'\,;\;\; f_{13}=\gamma(f_{13}'+vg_3')\,;\;\; f_{21}=\gamma(f_{21}'+\frac{\beta}{c}\mathfrak{P}_2') \\
\mathfrak{P}_1=\gamma^2[\mathfrak{P}_1'+v^2g_1'+v(f_{11}'+u')]\,;\;\; \mathfrak{P}_2=\gamma(\mathfrak{P}_2'+vf_{21}')\,;\;\; \mathfrak{P}_3=\gamma(\mathfrak{P}_3'+vf_{31}') \\
g_1=\gamma^2[g_1'+\frac{\beta^2}{c^2}\mathfrak{P}_1'+\frac{\beta}{c}(u'+f_{11}')]\,;\;\; g_2=\gamma(g_2'+\frac{\beta}{c}f_{12}')\,;\;\; g_3=\gamma(g_3'+\frac{\beta}{c}f_{13}') \\
\qquad u=\gamma^2[u'+\beta^2f_{11}'+\frac{\beta}{c}(\mathfrak{P}_1'+c^2g_1')].
\end{array}\right\} \tag{12a}$$

Here the x-axis is taken along $\mathbf{v}$, the velocity of S' relative to S. (The reader can condense these formulae into a more convenient shape by using vectors and the stretcher ϵ.) If there is, from the S'-point of view, no flux of energy and no momentum, then u and the stress-components become as in (14a) of Chap. IX.; we obtain also the same S-momentum as before, *i.e.*

$$g_1=\frac{\gamma^2v}{c^2}(u'+f_{11}'),\;\; g_2=\frac{\gamma v}{c^2}f_{12}',\;\; g_3=\frac{\gamma v}{c^2}f_{13}',$$

whereas

$$\mathfrak{P}_1=\frac{\gamma^2v}{c^2}(u'+f_{11}'),\;\; \mathfrak{P}_2=\gamma vf_{21}',\;\; \mathfrak{P}_3=\gamma vf_{31}'.$$

Thus, $c^2\mathbf{g}$ and $\mathfrak{P}$ may still differ from one another. But if the stress in S' is self-conjugate, the two vectors *become equal*, and the formulae of Chap. IX. are again obtained. In the case of electrodynamics, for instance, the latter condition, $f_{\iota\kappa}'=f_{\kappa\iota}'$, will be

seen to hold for any electromagnetic field, if S' is attached to the ponderable medium; and the condition of vanishing g' and $\mathfrak{P}'$ will be satisfied in the case of a purely electrostatic, or a purely magnetostatic field.

With $f_{\iota\kappa}'=f_{\kappa\iota}'$ alone, we have, from (12*a*), the interesting relation

$$\mathfrak{P}-c^2\mathbf{g}=\frac{\gamma}{\epsilon}[\mathfrak{P}'-c^2\mathbf{g}'], \tag{13}$$

which will hold for *any* electromagnetic field, provided that S' is the rest-system of the ponderable medium.

But let us return to our chief subject. After what has been said we could either employ the form (10) of the force-quaternion, and would then have to prove that $\mathfrak{S}$ is transformed according to (12), or we can proceed by satisfying our three requirements in their original forms (β), (γ), and (α). The two ways are wholly equivalent to one another. Minkowski chooses the former: he constructs $\mathfrak{S}$ in a manner that ensures by itself the validity of (12), subjects it to the operation lor, and develops the resulting four-vector.* We shall take the latter way, which the reader may find easier to follow. Thus, we shall first construct F so that it should be a physical quaternion, and then find the corresponding expressions for the energy, stress, etc., according to (β), (γ), aided, of course, by the electromagnetic equations (1), (2).

The first step to be taken is suggested by analogy with the construction of the fundamental electronic force-expression (cf. p. 220). We know that

$$C=\iota\rho+\frac{\mathbf{I}}{c}\simeq q,$$

and that $\mathbf{L}=\mathfrak{M}-\iota\mathbf{E}$ is a left-handed bivector. Therefore, $C\mathbf{L}\sim q$. Similarly, $\mathbf{R}=\mathfrak{M}+\iota\mathbf{E}$ being a right-handed bivector, we have $\mathbf{R}C\sim q$. The difference of both products has also the structure of q, and thus is again a physical quaternion, and can be used as far as (α) is concerned. Try, therefore, to satisfy the remaining requirements of the problem by putting

$$F=\tfrac{1}{2}[C\mathbf{L}-\mathbf{R}C]. \tag{14a}$$

This will turn out to represent the whole force-quaternion in the case of a *homogeneous* medium, and will, for heterogeneous media,

* See **Note 3** at the end of the chapter.

be easily supplemented by another physical quaternion involving the variations of K, μ. Develop the right-hand side of (14a). Then the vector part will give *the ponderomotive force*,

$$\mathbf{P} = \rho\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{I}\mathfrak{M}, \tag{15a}$$

and the scalar part will lead to

$$(\mathbf{Pv}) + J = (\mathbf{EI}). \tag{16a}$$

Eliminate $(\mathbf{EI})$ from these two equations and remember that $\mathbf{I} = \rho\mathbf{v} + \mathfrak{I}$. Then the result will be

$$J = \frac{1}{\rho}(\mathbf{P}\mathfrak{I}) = (\mathfrak{I}\mathbf{E}) + \frac{1}{c}(\mathfrak{I}\mathrm{V}\mathbf{v}\mathfrak{M}),$$

giving for *Joule's heat* the expression

$$J = (\mathfrak{I}\mathbf{E}^{\times}). \tag{17}$$

Thus far (β) and (γ) have not yet been employed. Now take account of these conditions, beginning with the latter. This gives, by (16a),

$$-(\mathbf{EI}) = \frac{\partial u}{\partial t} + \operatorname{div}\mathfrak{P}.$$

Now, by the electromagnetic differential equations (1),

$$-(\mathbf{EI}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\mathfrak{E} + \mathbf{M}\mathfrak{M}) + c\,.\operatorname{div}\mathrm{V}\mathbf{EM}.$$

Thus (γ), the principle of conservation of energy, is satisfied if the density of *electromagnetic energy* is taken to be

$$u = \frac{1}{2}(\mathbf{E}\mathfrak{E} + \mathbf{M}\mathfrak{M}), \tag{18}$$

and the *flux of energy*, from the standpoint of the observing system,*

$$\mathfrak{P} = c\mathrm{V}\mathbf{EM}. \tag{19}$$

The addition of an arbitrary sourceless (solenoidal) flux, as well as of an invariable u-term, would be irrelevant.

Lastly, to represent the ponderomotive force in the form required by (β), introduce in (15a) the first pair of equations (1). Then

$$\mathbf{P} = \mathbf{E}\operatorname{div}\mathfrak{E} - \mathrm{V}\mathfrak{M}\operatorname{curl}\mathbf{M} - \frac{1}{c}\mathrm{V}\frac{\partial\mathfrak{E}}{\partial t}\mathfrak{M}.$$

* Remember that $\partial/\partial t$ is the symbol of local time-variation in the observing system S.

Using the second pair of equations (1), and writing, for the moment,

$$\mathbf{A} = \mathbf{E} \operatorname{div} \mathfrak{E} - V\mathfrak{E} \operatorname{curl} \mathbf{E} + \mathbf{M} \operatorname{div} \mathfrak{M} - V\mathfrak{M} \operatorname{curl} \mathbf{M},$$

we have

$$\mathbf{P} = \mathbf{A} - \frac{1}{c}\frac{\partial}{\partial t} V\mathfrak{E}\mathfrak{M}.$$

This gives, first of all, for *the electromagnetic momentum* per unit volume,

$$\mathbf{g} = \frac{1}{c} V\mathfrak{E}\mathfrak{M}, \tag{20}$$

and what remains to be shown is that the vector sum **A**, familiar from the Maxwellian theory, is of the form $-\nabla f$. Now, this is exactly the case, provided that K and μ, involved in (2), are *constant* throughout the medium. In fact, take for *the electromagnetic stress* the familiar expression

$$\mathbf{f}_n = u\mathbf{n} - \mathbf{E}(\mathfrak{E}\mathbf{n}) - \mathbf{M}(\mathfrak{M}\mathbf{n}), \tag{21}$$

where u is as in (18). Then, remembering that ∇f is used as shorthand for $\partial \mathbf{f}_1/\partial x + \partial \mathbf{f}_2/\partial y + \partial \mathbf{f}_3/\partial z$,

$$-\nabla f = -\nabla u + \frac{\partial}{\partial x}[(\mathfrak{E}\mathbf{i})\mathbf{E} + (\mathfrak{M}\mathbf{i})\mathbf{M}] + \frac{\partial}{\partial y}\cdots + \frac{\partial}{\partial z}\cdots$$

$$= \mathbf{E} \operatorname{div} \mathfrak{E} + \mathbf{M} \operatorname{div} \mathfrak{M} - \nabla u + (\mathfrak{E} \,.\, \nabla)\mathbf{E} + (\mathfrak{M} \,.\, \nabla)\mathbf{M}.$$

On the other hand, we have

$$V\mathfrak{E} \operatorname{curl} \mathbf{E} = V\mathfrak{E} \,.\, V\nabla\mathbf{E} = \nabla(\mathbf{E} \,.\, \mathfrak{E}) - (\mathfrak{E} \,.\, \nabla)\mathbf{E}$$

(where the dot stops ∇'s differentiating action), and a similar expression for the last term of **A**. Thus,

$$-\nabla f = \mathbf{A} + \mathbf{N},$$

where

$$\mathbf{N} = \nabla(\mathbf{E} \,.\, \mathfrak{E} + \mathbf{M} \,.\, \mathfrak{M}) - \nabla u,$$

i.e.

$$\mathbf{N} = \tfrac{1}{2}\nabla(\mathbf{E} \,.\, \mathfrak{E} + \mathbf{M} \,.\, \mathfrak{M} - \mathfrak{E} \,.\, \mathbf{E} - \mathfrak{M} \,.\, \mathbf{M}).$$

To prevent a possible misunderstanding, we may add that this is a vector whose components are

$$N_1 = \tfrac{1}{2}\left(\mathfrak{E}\frac{\partial \mathbf{E}}{\partial x} + \mathfrak{M}\frac{\partial \mathbf{M}}{\partial x} - \mathbf{E}\frac{\partial \mathfrak{E}}{\partial x} - \mathbf{M}\frac{\partial \mathfrak{M}}{\partial x}\right), \quad \text{etc.}$$

Now, returning to the relations $\mathfrak{E}^\times = K\mathbf{E}^\times$, $\mathfrak{M}^\times = \mu\mathbf{M}^\times$, and effecting a transformation, the details of which will be found in Minkowski's

paper,* the reader will verify that the above vector is identical with

$$\mathbf{N} = -\tfrac{1}{2}(\mathrm{T}\eta)^2 . \nabla K - \tfrac{1}{2}(\mathrm{T}\zeta)^2 . \nabla\mu,$$

where the quaternions η and ζ are as in (B), p. 265. In the case of homogeneity, therefore, **N** vanishes, and we have $\mathbf{A} = -\nabla f$, so that the condition (β) is satisfied, with the above stress and momentum, by taking (15a) for the ponderomotive force, that is (14a) for the force-quaternion.

In the more general case of a *heterogeneous* medium we have only to supplement our original **P** by the vector **N**, and consequently to add to our original F the quaternion

$$-\tfrac{1}{2}(\mathrm{T}\eta)^2 . DK - \tfrac{1}{2}(\mathrm{T}\zeta)^2 . D\mu, \tag{c}$$

which, like that F itself, is $\simeq q$, since $\mathrm{T}\eta$ and $\mathrm{T}\zeta$, being the tensors of physical quaternions, are invariant with respect to the Lorentz transformation.

Thus we shall have, as a generalization of (14a),

$$F = \tfrac{1}{2}[C\mathbf{L} - \mathbf{R}C] - \tfrac{1}{2}(\mathrm{T}\eta)^2 . DK - \tfrac{1}{2}(\mathrm{T}\zeta)^2 . D\mu, \tag{14}$$

which splits into

$$\mathbf{P} = \rho\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{I}\mathfrak{M} - \tfrac{1}{2}(\mathrm{T}\eta)^2 . \nabla K - \tfrac{1}{2}(\mathrm{T}\zeta)^2 . \nabla\mu \tag{15}$$

and

$$(\mathbf{P}\mathbf{v}) + J = (\mathbf{E}\mathbf{I}) + \tfrac{1}{2}(\mathrm{T}\eta)^2\frac{\partial K}{\partial t} + \tfrac{1}{2}(\mathrm{T}\zeta)^2\frac{\partial\mu}{\partial t}. \tag{16}$$

All requirements being now satisfied, with the above values of density and flux of energy, and of stress and momentum, the only thing to be still revised on account of the heterogeneity of the medium is the Joulean waste. Now, proceeding as before, we obtain at once, from (15) and (16),

$$J = (\mathfrak{I}\mathbf{E}^{\times}) + \tfrac{1}{2}(\mathrm{T}\eta)^2\frac{dK}{dt} + \tfrac{1}{2}(\mathrm{T}\,\zeta)^2\frac{d\mu}{dt},$$

where

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} + (\mathbf{v}\nabla)K = \frac{1}{\gamma}\frac{\partial K}{\partial t'},$$

and similarly, $d\mu/dt = \partial\mu/\gamma\partial t'$. Thus, if there is, from the standpoint of the rest-system, *no time-variation* of K, μ, we re-obtain

* *Grundgleichungen*, formula (92), in which the last term, due to the changes of the velocity of motion, is to be omitted.

the previous value, (17), of Joule's heat. Under these circumstances we have also

$$\nabla K = \epsilon \nabla' K, \quad \nabla \mu = \epsilon \nabla' \mu,$$

which values can be substituted in the last two terms of (15). To resume:

The force-quaternion

$$F = \tfrac{1}{2}[C\mathbf{L} - \mathbf{R}C] - \tfrac{1}{2}(\mathrm{T}\eta)^2 . DK - \tfrac{1}{2}(\mathrm{T}\zeta)^2 . D\mu, \qquad (14)$$

being a physical quaternion, satisfies the fundamental relativistic requirements and, at the same time, the so-called *principle of momentum*,

$$\mathbf{P} = -\nabla f - \partial \mathbf{g}/\partial t, \qquad (\beta)$$

and the *principle of conservation of energy*,

$$(\mathbf{P}\mathbf{v}) + J + \frac{\partial u}{\partial t} + \operatorname{div} \mathfrak{P} = 0. \qquad (\gamma)$$

It gives for *the ponderomotive force*, per unit volume of an isotropic medium,

$$\mathbf{P} = \rho\mathbf{E} + \frac{1}{c}\mathrm{V}\mathbf{I}\mathfrak{M} - \tfrac{1}{2}(T\eta)^2 . \nabla K - \tfrac{1}{2}(T\zeta)^2 . \nabla\mu, \qquad (15)$$

and for the *Joulean waste* (with $\partial K/\partial t' = \partial\mu/\partial t' = 0$):

$$J = (\mathfrak{I}\mathbf{E}^{\times}), \qquad (17)$$

where $\mathfrak{I} = \mathbf{I} - \rho\mathbf{v}$ is the conduction current. The corresponding auxiliary magnitudes are as follows:

The density of *electromagnetic energy*

$$u = \tfrac{1}{2}(\mathbf{E}\mathfrak{E} + \mathbf{M}\mathfrak{M}), \qquad (18)$$

the flux of energy

$$\mathfrak{P} = c\mathrm{V}\mathbf{E}\mathbf{M}, \qquad (19)$$

the density of *electromagnetic momentum*

$$\mathbf{g} = \frac{1}{c}\mathrm{V}\mathfrak{E}\mathfrak{M}, \qquad (20)$$

and, finally, *the electromagnetic stress*

$$\mathbf{f}_n = u\mathbf{n} - \mathbf{E}(\mathfrak{E}\mathbf{n}) - \mathbf{M}(\mathfrak{M}\mathbf{n}). \qquad (21)$$

The physical quaternions η and ζ are, as on p. 265,

$$\eta = \gamma\left[\frac{\iota}{c}(\mathbf{E}^{\times}\mathbf{v}) + \mathbf{E}^{\times}\right], \quad \zeta = \gamma\left[\frac{\iota}{c}(\mathbf{M}^{\times}\mathbf{v}) + \mathbf{M}^{\times}\right]. \qquad (22)$$

Our F is the quaternionic equivalent of Minkowski's four-vector K (*loc. cit.*, § 14, for constant **v**, of course), and so also our stress-components, etc., are identical with the sixteen constituents of Minkowski's matrix $-S$. The difference between the theory here proposed and that given by Minkowski, is this:—What Minkowski considers as the ponderomotive force is the vector, *not of F itself* but, of

$$F_{\text{Mnk}} = \tfrac{1}{2}\left[F + \frac{1}{c^2} Y F_c Y\right],$$

i.e. of that part of the four-vector F, which is *normal* to the four-velocity Y. Thus Minkowski's ponderomotive force is not of the form (β), though it becomes so in the rest-system. The reason why Minkowski proposed for the four-force the above part of F, instead of the whole F, is to be sought for in his dynamics, according to which the 'moving' four-force had always to be normal to the particle's world-line. This corresponded to the assumption of a constant rest-mass. But in general, as has been explained in Chapter IX., the four-force does not necessarily bear that relation to the world-line, and it will not do so whenever there is heat supply or heat generation. Now, this being exactly the case with an electric conductor, we had to abandon the Minkowskian condition of orthogonality. And, in connexion with this, Joule's heat has, from the beginning, been embodied into our force-quaternion along with the activity of the force, such a procedure being directly suggested by the dynamical considerations of Chapter IX. If there is no conductivity, and therefore also no Joulean waste, then the four-force represented by (14) is, in fact, normal to the world-line of the body, *i.e.*

$$S Y_c F = 0, \quad \text{for} \quad \sigma = 0,$$

as is proved in **Note 4** at the end of the chapter. But in a conducting body this property does not hold.

We will content ourselves here with the above modification of Minkowskian electrodynamics, which, besides fulfilling the stated requirements, is also in complete agreement with what is known from experience. It is true that Einstein and Laub * missed in the ponderomotive force of the above kind a term due to displacement-current (uniform with those due to conduction- and

* *Ann. der Physik*, Vol. XXVI., 1908, p. 541.

convection-current, contained in $\mathrm{V}\mathbf{I}\mathfrak{M}/c$). But, since nobody has ever observed such an action of the magnetic field, this can hardly be considered as a serious objection. In order to obtain the desired term, Einstein and Laub recurred to the electron theory, and since in doing so they thought necessary to limit themselves to the case of stationary bodies, their electrodynamics is of no particular interest from the relativistic point of view. Abraham's ponderomotive force contains, in addition to (15), several other terms, and among these the displacement-current term. His electrodynamics of moving bodies,* as has already been mentioned, is based upon the assumption of the particular relation $\mathbf{g} = \mathfrak{P}/c^2$, borrowed from the vacuum-equations. The desire to retain this relation throughout the theory makes Abraham's formulae considerably more complicated than the above ones. His expression for the Joulean waste, subject to the same conditions for K, μ, is the same as above, but those for the density and flux of energy, and consequently also for the momentum, are less simple.

The set of electrodynamical formulae given above is best characterized by saying that, while satisfying the principle of relativity and the principle of conservation of energy, it gives for the rest-system the ponderomotive force

$$\mathbf{P}' = -\nabla' f'_{\mathrm{Mxw}} - \frac{1}{c}\frac{\partial}{\partial t'}\mathrm{V}\mathfrak{E}'\mathfrak{M}'. \tag{23}$$

In fact, f, as given by (21), reduces in S' to the Maxwellian stress-operator,

$$f' = f'_{\mathrm{Mxw}} = \tfrac{1}{2}(KE'^2 + \mu M'^2) - K\mathbf{E}'(\mathbf{E}' - \mu\mathbf{M}'(\mathbf{M}' \;.$$

It will be remembered that Maxwell's stress, taken by itself, would give, in absence of electric charge and of ponderable matter, the ponderomotive force

$$\frac{1}{c}\frac{\partial}{\partial t'}\mathrm{V}\mathbf{E}'\mathbf{M}',$$

and this 'force on the free aether' is just balanced by the second term in (23). See p. 48. With the exception of this obviously desirable supplementary term everything is as in Maxwell's electrodynamics of stationary bodies. Thus, (15), the developed form of the ponderomotive force, becomes in the rest-system, by (A),

$$\mathbf{P}' = \rho'\mathbf{E}' + \frac{1}{c}\mathrm{V}\mathfrak{I}'\mathfrak{M}' - \tfrac{1}{2}E'^2 . \nabla' K - \tfrac{1}{2}M'^2 . \nabla'\mu,$$

* Cf. footnote on p. 240.

where $\mathfrak{I}' = \mathbf{I}'$ is the conduction current, and each of the four terms has its old familiar form and meaning. Again, by (17) and (A), the Joulean waste takes its usual form,

$$J' = (\mathbf{I}'\mathbf{E}') = \sigma E'^2,$$

while (18) and (19) give at once the Maxwellian density of electromagnetic energy, and the familiar Poynting vector for the flux of energy,

$$u' = \tfrac{1}{2}(KE'^2 + \mu M'^2)\,; \quad \mathfrak{P}' = c\mathrm{V}\mathbf{E}'\mathbf{M}'.$$

The transformation formula for Joule's heat is easily obtained. In fact, since F, defined by (9), is a physical quaternion, and since

$$F' = \frac{\iota}{c}J' + \mathbf{P}',$$

we have at once, writing P_1 for the longitudinal component of the force,

$$J + vP_1 = \gamma(J' + vP_1')$$

and

$$P_1 = \gamma(P_1' + vJ'/c^2),$$

whence, by subtraction,

$$J = \gamma(1 - \beta^2)J' = \frac{1}{\gamma}J'.$$

Consequently, if dS' be any volume-element of the body, and dS its correspondent,

$$J\,dS = \frac{1}{\gamma^2}J'dS',$$

in complete agreement with (29), Chap. IX.

The electromagnetic momentum bears, in the rest-system, a simple relation to the energy flux. In fact, by (20),

$$\mathbf{g}' = \frac{K\mu}{c}\mathrm{V}\mathbf{E}'\mathbf{M}' = \frac{c}{\mathfrak{v}'^2}\mathrm{V}\mathbf{E}'\mathbf{M}',$$

where, dispersion being disregarded, $\mathfrak{v}'$ is the velocity of propagation of disturbances, as estimated by the S'-observers. Hence, instead of Planck's relation, we have

$$\mathbf{g}' = \frac{1}{\mathfrak{v}'^2}\mathfrak{P}', \tag{24}$$

so that, in a stationary ponderable medium, $\mathfrak{v}'$ takes the place of c. And since $\mathfrak{v}'$ plays in such a medium just the same part as the

critical velocity in empty space, it seems quite natural that (24) should replace the relation which holds good in the absence of matter. The stress in S' being self-conjugate, our previous equation (13) can be applied, so that, in general,

$$\mathfrak{P} - c^2\mathbf{g} = (1 - n'^2)\gamma\epsilon^{-1}\mathfrak{P}',$$

where n' is the refractive index of the medium. If, therefore, n' differs at all from unity, we have $\mathfrak{P} \neq c^2\mathbf{g}$, unless there is in the rest-system no Poynting flux.

Finally, notice that if $K = \mu = 1$, and $\sigma = 0$, the ponderomotive force (15) coincides with that of the electron theory. And the same thing is true of the above expressions for stress, density and flux of energy, and momentum. So also were the vacuum-equations contained, as a limiting case, in Minkowski's electromagnetic differential equations for moving bodies.

NOTES TO CHAPTER X.

Note 1 (to page 266). Let the quaternions a and b represent a pair of four-vectors. Then the component of the four-vector a taken along the four-vector b (cf. p. 148) will be represented by

$$(\mathrm{T}b)^{-1}.\, \mathrm{S}a_c b,$$

and, consequently, the part of a normal to b, in both size and direction, by

$$a_n = a - \frac{b\mathrm{S}a_c b}{(\mathrm{T}b)^2}\cdot$$

Now, $\mathrm{S}a_c b = \frac{1}{2}[b_c a + a_c b]$, and $bb_c = (\mathrm{T}b)^2$. Hence

$$a_n = \frac{1}{2}\left[a - \frac{ba_c b}{(\mathrm{T}b)^2}\right],$$

which is the required expression.

Note 2 (to page 273). It will be enough to consider here the case of plane waves, propagated along $\mathbf{v}$, in a non-conducting medium, carrying no charge, so that $\mathbf{I} = 0$.

As in a previous Note (p. 59), take $\mathbf{E}$, $\mathfrak{E}$, etc., proportional to an exponential function of the argument

$$g(x - \mathfrak{b}t),$$

where g is an imaginary constant, and $\mathfrak{b}$ the velocity of propagation, as estimated from the S-point of view (or else consider a wave of

discontinuity). Then the equations (1) and (2) will give, by (A), p. 265, and since $\mathbf{v}=c\beta\mathbf{i}$,

$$-\frac{\mathfrak{v}}{c}\mathfrak{E}=\mathrm{V}\mathbf{i}\mathbf{M},\quad \frac{\mathfrak{v}}{c}\mathfrak{M}=\mathrm{V}\mathbf{i}\mathbf{E}, \tag{a}$$

$$\mathfrak{E}+\beta\mathrm{V}\mathbf{i}\mathbf{M}=K[\mathbf{E}+\beta\mathrm{V}\mathbf{i}\mathfrak{M}], \tag{b}$$

$$\mathfrak{M}-\beta\mathrm{V}\mathbf{i}\mathbf{E}=\mu[\mathbf{M}-\beta\mathrm{V}\mathbf{i}\mathfrak{E}], \tag{c}$$

the solenoidal conditions $(\mathfrak{E}\mathbf{i})=(\mathfrak{M}\mathbf{i})=0$ being already satisfied by (*a*). Next, introduce (*a*) into (*b*), (*c*); then

$$\mathfrak{E}\left[1-\beta\frac{\mathfrak{v}}{c}\right]=K\left[1-\beta\frac{c}{\mathfrak{v}}\right]\mathbf{E},$$

$$\mathfrak{M}\left[1-\beta\frac{\mathfrak{v}}{c}\right]=\mu\left[1-\beta\frac{c}{\mathfrak{v}}\right]\mathbf{M},$$

showing that $\mathbf{E}$ and $\mathbf{M}$ are again transversal. Use the latter relations in (*a*), eliminate either $\mathbf{E}$ or $\mathbf{M}$, and remember that

$$\mathfrak{v}'=c/\sqrt{K\mu}.$$

Then the result will be

$$\frac{\mathfrak{v}-v}{1-v\mathfrak{v}/c^2}=\mathfrak{v}',$$

whence

$$\mathfrak{v}=\frac{\mathfrak{v}'+v}{1+v\mathfrak{v}'/c^2}.$$

Thus $\mathfrak{v}$ is obtained from $\mathfrak{v}'$ and v by Einstein's addition theorem of velocities, and this, as we saw on p. 172, gives the Fresnelian value for the dragging coefficient.

Note 3 (to page 279). Let h and H, as on p. 264, be the alternating matrices equivalent to the electromagnetic bivectors $\mathfrak{L}$ and $\mathbf{L}$ respectively, *i.e.*

$$h_{23}=M_1,\qquad h_{31}=M_2,\qquad h_{12}=M_3$$
$$h_{14}=-\iota\mathfrak{E}_1,\quad h_{24}=-\iota\mathfrak{E}_2,\quad h_{34}=-\iota\mathfrak{E}_3$$

and

$$H_{23}=\mathfrak{M}_1,\qquad H_{31}=\mathfrak{M}_2,\qquad H_{12}=\mathfrak{M}_3$$
$$H_{14}=-\iota E_1,\quad H_{24}=-\iota E_2,\quad H_{34}=-\iota E_3.$$

Both of these matrices reduce, for $K=\mu=1$, to the matrix h of **Note 2** to Chap. IX.

Minkowski begins by constructing the product of h into H. Since each of the factors is transformed by $\bar{A}(\ )A$, the same will be true of their product, which will be a matrix of 4×4 constituents. Now similarly as on p. 259, the reader will find

$$-hH=\mathfrak{S}+\lambda, \tag{a}$$

where $\mathfrak{S}$ is the matrix of the present chapter, whose constituents are exactly those given by (18) till (21), and

$$\lambda=\tfrac{1}{2}(\mathbf{M}\mathfrak{M}-\mathbf{E}\mathfrak{E}), \tag{b}$$

or what in the rest-system becomes 'the Lagrangian function.' (It may be mentioned, for the sake of comparison with Minkowski's paper, that our h, H, λ, $\mathfrak{S}$, F are his f, F, L, $-S$, K respectively.) Similarly, h^* and H^* being the dual matrices,

$$-H^*h^* = -\mathfrak{S} + \lambda. \tag{c}$$

By (a) and (c), λ is an invariant, and $\mathfrak{S}$ is transformed by $\bar{A}(\ )A$, so that

$$F = -\operatorname{lor}\mathfrak{S}$$

is a genuine four-vector (or physical quaternion). The latter becomes, by (a), (c) and (b),

$$F = \operatorname{lor} h\,.\,H - \operatorname{lor} H^*\,.\,h^* + \mathrm{N}, \tag{d}$$

where the dots act as separators and N is the four-vector written in quaternionic form under (C), p. 282.

Next, using in (d) the differential equations of the field, $\operatorname{lor} h = -s$ $\operatorname{lor} H^* = 0$, Minkowski obtains

$$F = sH + \mathrm{N}, \tag{e}$$

where s is the current-matrix, represented in this chapter by the quaternion C. Minkowski's ponderomotive four-force is *the part* of (e) normal to the four-velocity. Our force-quaternion (14) is the quaternionic equivalent of *the whole* four-vector (e).

Notice that (b) can be written, in terms of the electromagnetic bivectors,

$$\lambda = -\tfrac{1}{2}\mathrm{S}\mathbf{L}\mathfrak{L},$$

whence the invariance of λ is seen immediately.

Note 4 (to page 284). For a *non-conducting* medium, the current-quaternion becomes

$$C = \rho\left[\iota + \frac{1}{c}\mathbf{v}\right] = \frac{\rho}{c\gamma}Y,$$

and therefore, the force-quaternion (14),

$$F = \frac{\rho}{\gamma}\eta - \tfrac{1}{2}(T\eta)^2.\,DK - \tfrac{1}{2}(T\zeta)^2.\,D\mu,$$

where $2c\eta = Y\mathbf{L} - \mathbf{R}\,Y$, as on p. 265. Hence,

$$\mathrm{S}\,Y_c\eta = 0.$$

Again, as regards the second term of F,

$$\frac{1}{\gamma}\mathrm{S}\,Y_cDK = \frac{\partial K}{\partial t} + (\mathbf{v}\nabla)K = \frac{dK}{dt} = 0,$$

since $\partial K/\partial t' = 0$, by assumption. Similarly for the last term of the force-quaternion. Thus $\mathrm{S}\,Y_cF = 0$, which was to be proved.

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